DE GRUYTER OPEN

Reports on Geodesy and Geoinformatics vol. 101/2016; pp. 70-81 DOI: 10.1515/rgg-2016-0023

COMPARISON OF ROBUST ESTIMATORS FOR LEVELING NETWORKS IN MONTE CARLO SIMULATIONS

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Abstract

We compared the method of least squares (LS), Pope's iterative data snooping (IDS) and Huber's M-estimator (HU) in realistic leveling networks, for which the heights or the vertical displacements of points are known. The study was conducted using the Monte Carlo simulation, in which one repeatedly generates sets of observations related to the measurement data, then calculates values of the estimators and, finally, assesses it with respect to the real coordinates. To simulate outliers we used popular mixture models with two or more normal distributions. It is shown that for small, strong networks robust methods IDS and HU are more accurate than LS, but for large, weak networks occurring in practice there is no significant difference between the considered methods in the accuracy of the solution.

Keywords: *leveling network, robust estimation, outliers, gross errors, internal reliability index*

1. Introduction

Sometimes measurements in leveling networks are subject to gross contaminations that significantly affect the estimators of heights or vertical displacements of points determined by the least squares method (LS). In order to reduce the impact of possible gross errors, the robust estimators are employed. The essence of these methods is to repeatedly use LS with a simultaneous updating weights of observations such that outliers have less impact on the resulting solution. In these iterations as outliers we consider observations, for which the calculated normalized residuals exceed a given critical value.

In literature, there are currently many different methods of robust estimation. For short introduction we refer to Knight and Wang (2009) and for broad overview to Huber and Ronchetti (2009). In particular we have, popular in geodesy, methods which combine LS with statistical testing for normalized residuals such as the Baarda's or the Pope's iterative data snooping (IDS). Moreover, there are M-estimators minimizing error functions which increase slower than a quadratic function used by LS. One of the most well-known methods in this group is Huber's estimator (HU).

In this work we compare accuracy of LS, IDS and HU methods in systematic, extensive Monte-Carlo simulations using four realistic leveling networks and popular models of gross errors that are mixtures of two or more normal distributions.

2. Linear model for geodetic networks

To find heights or displacements of points we will use the linear model:

$$\tilde{\mathbf{A}}\mathbf{x} + \tilde{\boldsymbol{\varepsilon}} = \tilde{\mathbf{y}}, \tag{1}$$

where $\tilde{\mathbf{A}}$ is a $n \times p$ design matrix, $\mathbf{x}$ is a vector of p parameters, $\tilde{\boldsymbol{\varepsilon}}$ is a vector of n random errors in measurements and $\tilde{\mathbf{y}}$ is a vector of n observations. We assume that $E(\tilde{\boldsymbol{\varepsilon}}) = 0$, $\mathrm{var}(\tilde{\boldsymbol{\varepsilon}}) = \sigma^2 \mathbf{P}^{-1}$, where $\mathbf{P}$ is a diagonal matrix of weights. The observations are height differences between benchmarks, which constitute the end points of leveling sections. Usually they are assigned weights inversely proportional to the number of stations: $p_i = n_i^{-1}$, where n_i - number of stations for *i*-th leveling section.

After multiplying both sides of the equation (1) by $\mathbf{P}^{1/2}$, we obtain the homoscedastic linear model

$$\mathbf{A}\mathbf{x} + \boldsymbol{\varepsilon} = \mathbf{y} \tag{2}$$

where $\mathbf{A} = \mathbf{P}^{1/2}\tilde{\mathbf{A}},\ \boldsymbol{\varepsilon} = \mathbf{P}^{1/2}\tilde{\boldsymbol{\varepsilon}},\ \mathbf{y} = \mathbf{P}^{1/2}\tilde{\mathbf{y}}$, $E(\boldsymbol{\varepsilon}) = 0$ and $\mathrm{var}(\boldsymbol{\varepsilon}) = \sigma^2 \mathbf{I}_n$. Moreover $\sigma = \sigma_0 \sigma_f$, where σ_0 is a priori average error of observation and σ_f is a standard deviation of the gross error distribution with density f. The internal reliability index of the *i*-th observation is defined as $\sqrt{1 - \mathbf{H}_{ii}}$, where $\mathbf{H}_{ii}$ denotes the *i*-th diagonal element of the matrix $\mathbf{H} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$.

3. Estimators of the vector x

In our experiments we compare accuracy of the three following estimators of the vector $\mathbf{x}$:

1. $\hat{\mathbf{x}}_{LS}$ - the classical least squares estimator (LS) ,
2. $\hat{\mathbf{x}}_{IDS}$ - Pope's version of the iterative data snooping estimator (IDS) with unknown σ_0, that is LS in which the largest standardized residual is repeatedly removed from the dataset if it is greater than a critical value equals 2.5 as in Kwaśniak (2011) , and
3. $\hat{\mathbf{x}}_{HU}$ - the Huber estimator (HU) that is the well-known M-estimator, which minimize a spline of quadratic and linear functions in a knot equals 1.345 with a scale factor estimated by the Median Absolute Deviation. For details see

Knight and Wang (2009). These values are also default in the free statistical software R, which we used in simulations (R Core Team 2013).

As an error of estimation we compute in the *k*-th simulation, similarly as in Kwaśniak (2011), average L_2 distance:

$$errMET_k := \left\| \hat{\mathbf{x}}_{MET}^{(k)} - \mathbf{x} \right\| / \sqrt{p} \,, \tag{3}$$

where $MET \in \{LS, IDS, HU\}$

The result of the simulation for the method MET is given by the vector $\mathbf{errMET} = [errMET_1, ..., errMET_K]^T$.

4. Gross errors distributions

Let $\phi(x|\mu,\sigma)$ denote a density of normal distribution with mean μ and standard deviation σ. We consider three classical models of contamination of the typical measurement error defined by $\phi(x|0,1)$. All these models are given by mixtures of normal densities. In the first model we assume that the measurement error comes from *t* Student distribution with density $\tau(x|\nu)$, where $\nu > 2$ denotes degrees of freedom. Such model has been recently analysed in Koch and Kargoll (2013). Moreover, we consider the following models:

$$f_1(x|\alpha,\sigma_1) = (1-\alpha)\phi(x|0,1) + \alpha\phi(x|0,\sigma_1) \,, \tag{4}$$

$$f_2(x|\alpha,\mu_1,\sigma_1) = (1-\alpha)\phi(x|0,1) + \frac{\alpha}{2}\phi(x|-\mu_1,\sigma_1) + \frac{\alpha}{2}\phi(x|\mu_1,\sigma_1) \,, \tag{5}$$

where $0 \le \alpha \le 1$, $\mu_1 \ge 0$, $\sigma_1 \ge 1$. The model f_1 is popular in robust statistics (Huber and Ronchetti 2009), while the model f_2 is a modification of the gross error model described in Hekimoglu and Erenoglu (2007). Figure 1 shows graphs of densities used in simulations on the positive half-line, because they are symmetric functions. Fig. 1 (right) is an enlarged image of tails of densities. Table 1 shows the probabilities of gross errors in the relevant range of values given in the top row of the table. The last column provides the standard deviations of distributions.

5. Tested leveling networks

We compared accuracy of estimators LS, IDS and HU in the four leveling networks analysed in our department. Parameters of the networks such as structure of design matrices, inverse weights and observations are shown in the Appendix 1. Below we present short summaries and diagrams.

Sieć 1 (Prószyński and Kwaśniak 2002, p. 96) is a small network in which observations have high, internal reliability indices. In particular, the network consists of 10 points and 21 observations. Internal reliability indices for individual observations are within the range of 0.66 ÷ 0.83. We modified original weights of observations for consistency with the graph of Siec 1 shown in Figure 2a.

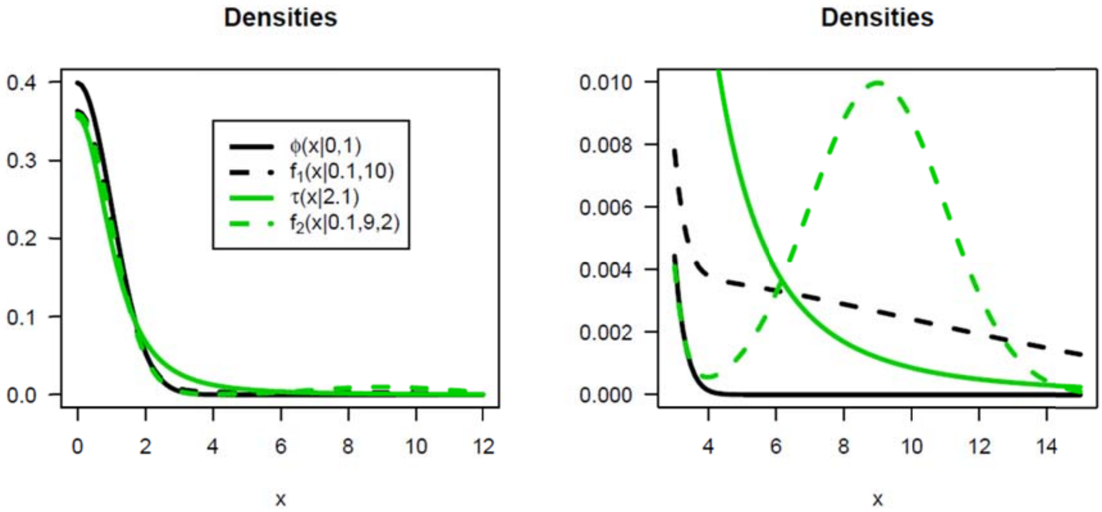

Fig. 1. Densities of the gross error distributions used in simulations.

Table 1. Probabilities of gross errors. In the column denoted „a-b" we show $P\{a \le |X| \le b\}$.

f \ $a-b$	0 – 3	3 – 6	6 – 9	9 – 12	12 – 15	15 – 18	σ_f
$\phi(x\|0, 1)$	0.99730	0.00270	0.00000	0.00000	0.00000	0.00000	1.00
$\tau(x\|2.1)$	0.91001	0.06626	0.01335	0.00466	0.00213	0.00114	4.58
$f_1(x\|0.1, 10)$	0.92115	0.02400	0.01804	0.01380	0.00965	0.00618	3.30
$f_2(x\|0.1, 9, 2)$	0.89770	0.00898	0.04332	0.04332	0.00655	0.00014	3.07

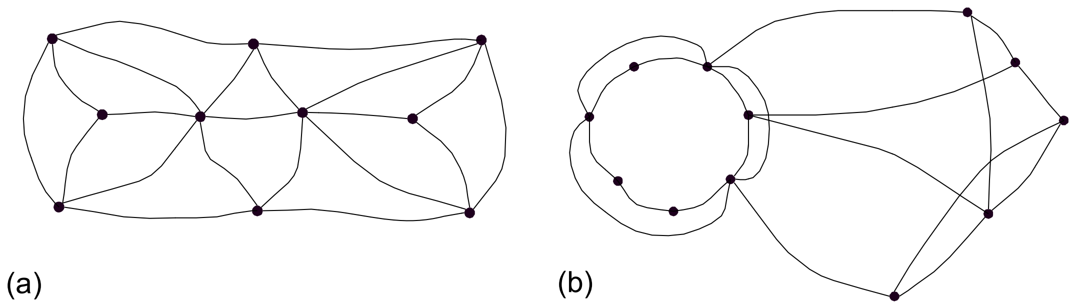

Fig. 2. Schemes of control leveling networs: (a) Sieć 1, (b) Sieć 2

Sieć 2 (Prószyński and Kwaśniak 2006, p. 92) consists of 12 points and 20 observations. The mean value of internal reliability index is equal to 0.67 (Fig. 2b).

The remaining two networks: Network *Łazienki* (Malarski et al. 2012) and Network *Św. Anna* (Malarski et al. 2010) are the result of real jobs in our department. These moderately large networks have more dispersed internal reliability indices than small networks. In the *Łazienki* network there are 53 points connected by 65 observations; in the *St. Anna* network 66 points and 80 observations, respectively. Diagrams of these networks are shown in Figure 3 and Figure 4.

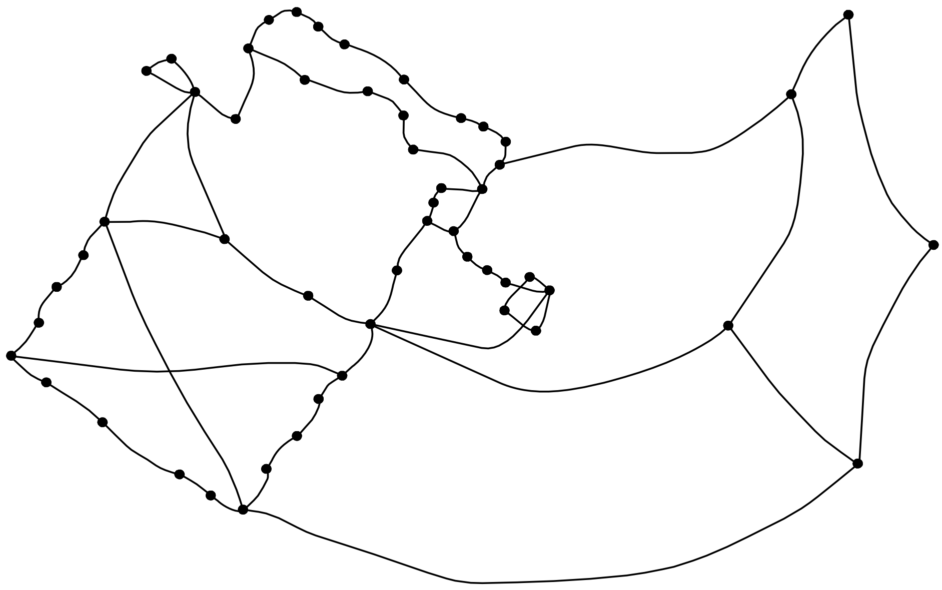

Fig. 3. Network *Łazienki*

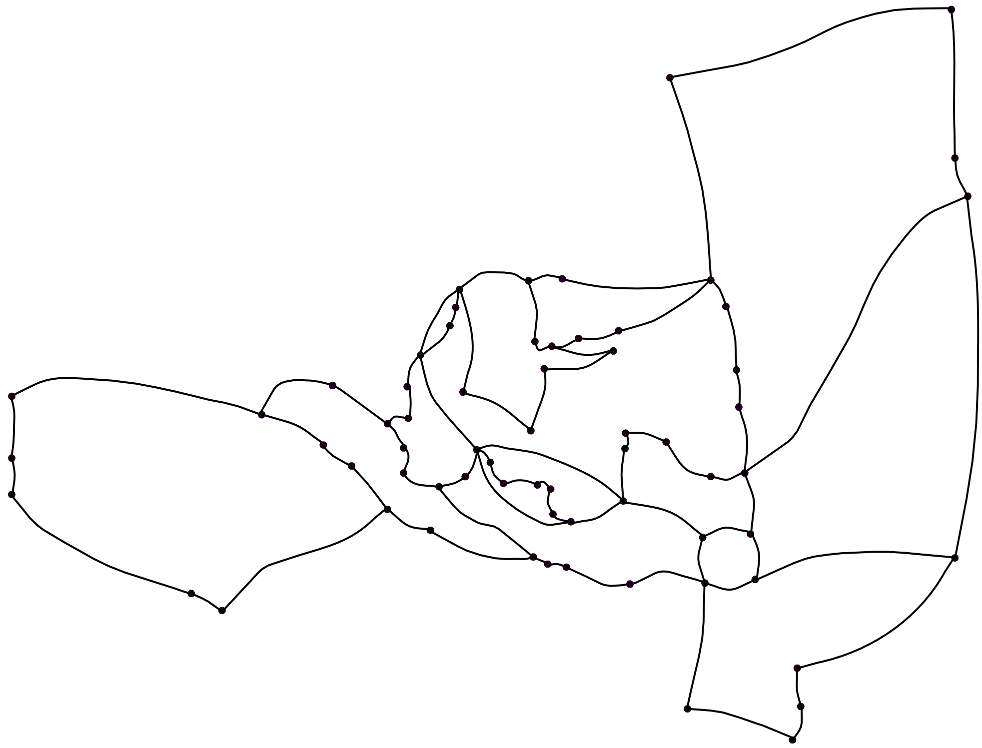

Fig. 4. Network *Św. Anna*

Table 2 contains parameters of the tested network: the number of observations n, the number of benchmarks p, the mean error of the typical measurement (given in mm) and quartiles of internal network reliability indices.

Table 2. Parameters of tested networks

Network	n	p	σ_0	Quartiles of internal reliability indices				
				min	Q_1	med.	Q_3	max
Sieć 1	21	10	0.08	0.66	0.72	0.75	0.81	0.83
Sieć 2	20	12	0.08	0.39	0.55	0.67	0.79	0.83
Łazienki	65	53	0.05	0.26	0.38	0.41	0.50	0.71
Św. Anna	80	66	0.05	0.22	0.35	0.38	0.49	0.79

6. Monte Carlo simulation

For each network $\mathbf{A}$, using original observations **y** and inverse weights given in the Appendix, we computed a solution $\mathbf{x}$ by LS and considered it in simulations as true. Indeed, these LS estimators were found to be correct by surveyors and customers, thus we can accept pairs $(\mathbf{A}, \mathbf{x})$ as realistic models. We also used original a priori average error of observation σ_0.

For given network $(\mathbf{A}, \mathbf{x}, \sigma_0)$ and given measurement error density f the following Monte Carlo simulation was repeated.

<u>for</u> $k = 1,..., K = 100000$
  $\boldsymbol{\varepsilon}^{(k)} =$ Generate (independently n times from f)
  $\mathbf{y}^{(k)} = \mathbf{A}\mathbf{x} + \sigma_0 \boldsymbol{\varepsilon}^{(k)}$
  Compute $\hat{\mathbf{x}}_{LS}^{(k)}$, $\hat{\mathbf{x}}_{IDS}^{(k)}$, $\hat{\mathbf{x}}_{HU}^{(k)}$ using ($\mathbf{y}^{(k)}, \mathbf{A}$)
  Compute $errLS_k$, $errIDS_k$, $errHU_k$
<u>end for</u>
Compare vectors $\mathbf{errLS}$, $\mathbf{errIDS}$, $\mathbf{errHU}$.

7. Comparison of estimators

Figures 5 and 6 present boxplots for data collected in $\mathbf{errLS}$, $\mathbf{errIDS}$, $\mathbf{errHU}$. Plots on the top of the figures correspond to the gross error density $f_1(x|0.1, 10)$, plots in the middle of figures are connected to $\tau(x|2.1)$, and on the bottom of the figures we can see results of simulations for $f_2(x|0.1, 9, 2)$. The scale on the vertical axis corresponds to the values of the estimation errors given in mm. On the plot, the y-coordinate of the lower line of the box is equal to the first quartile Q_1, but y-coordinate of the upper line of the box corresponds to third quartile Q_3. Thus the length of the box is equal to the interquartile range. Lines in the middle of the boxes denote medians. Sections shown by the dotted line, the so-called whiskers, show the spread of common values. It cannot be longer than one and a half of the interquartile range. Outliers are omitted on the plots.

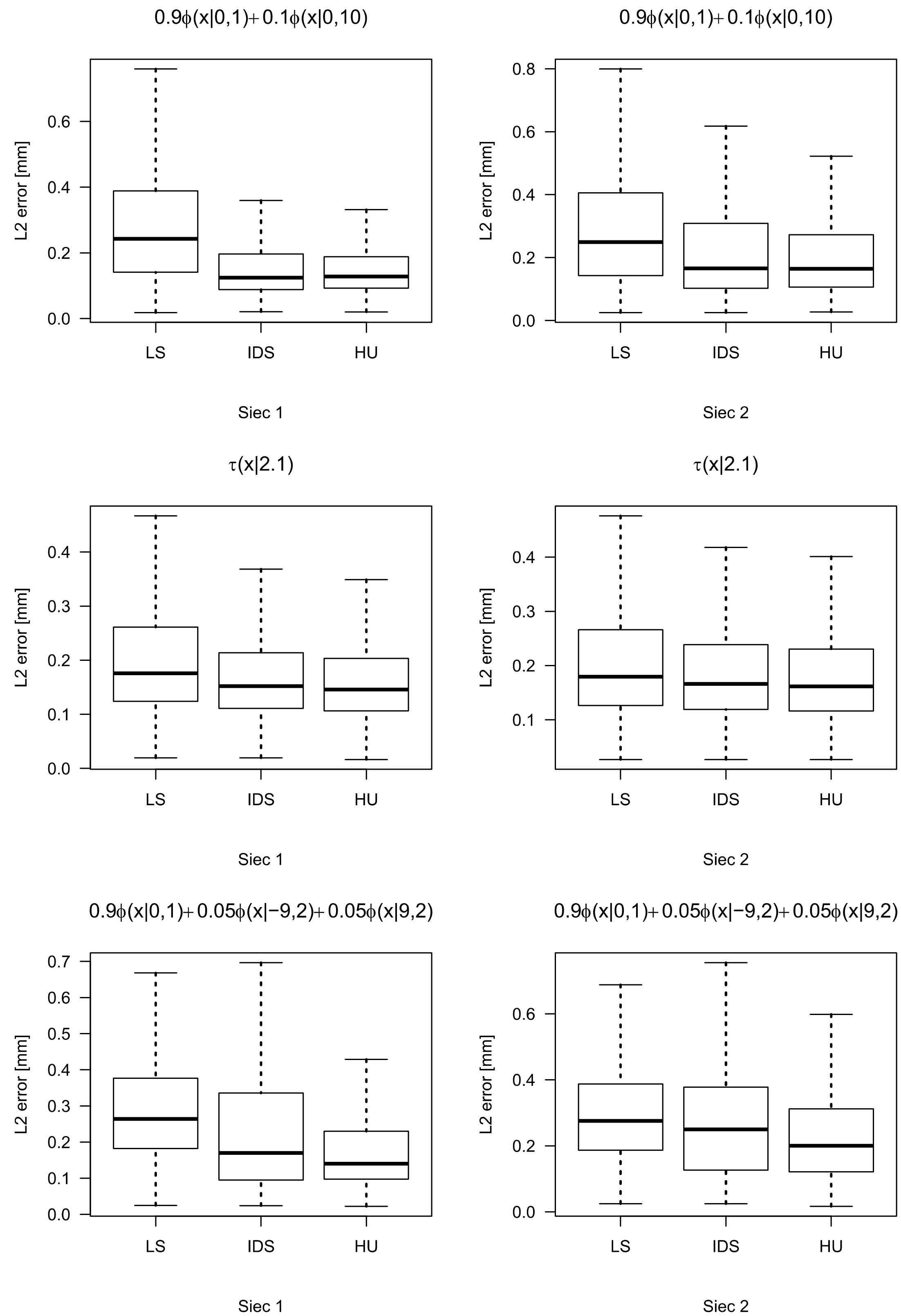

Fig. 5. Boxplots for estimation errors in small networks

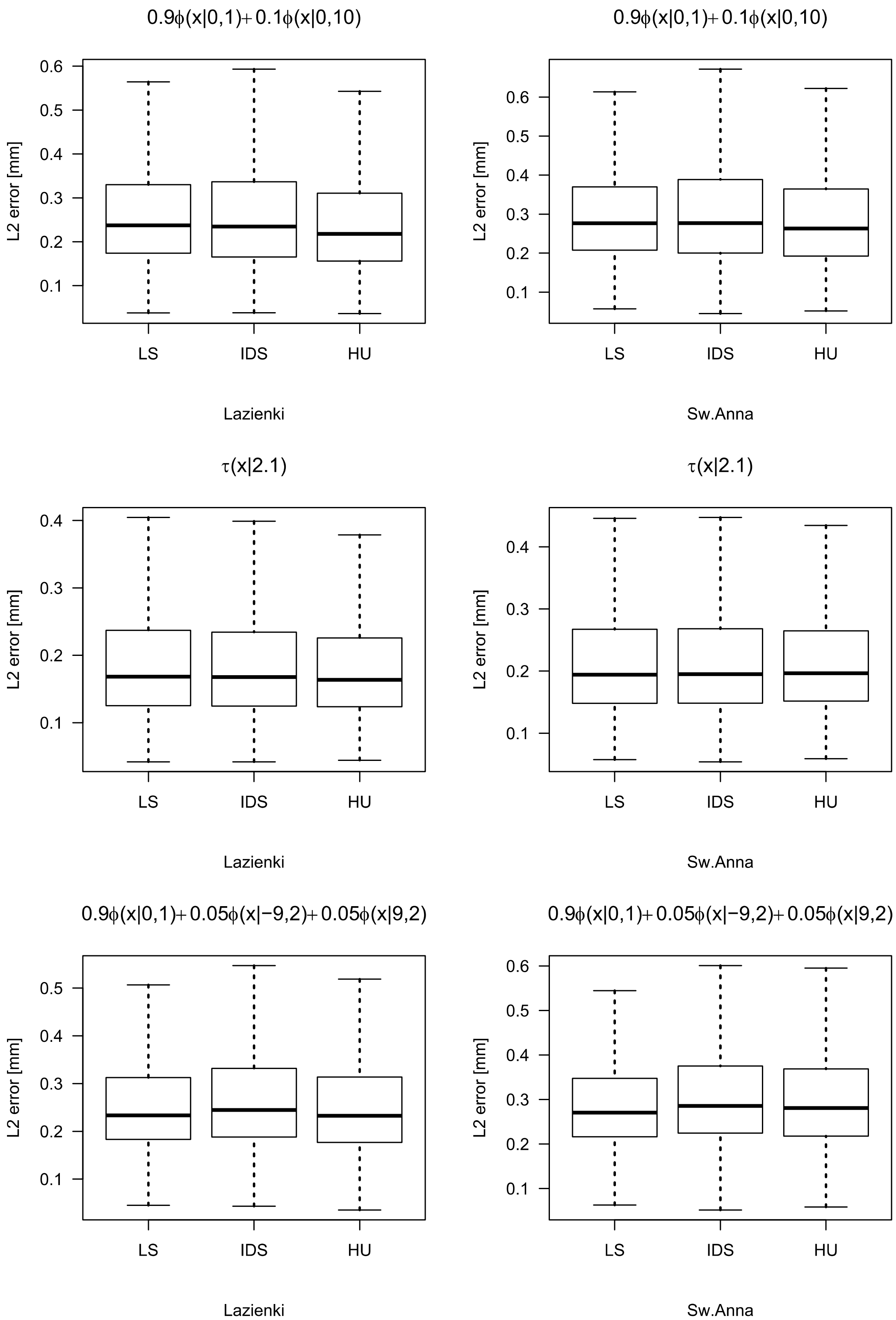

Fig. 6. Boxplots for estimation errors in networks *Łazienki* and *Św. Anna*

Let us observe that the distribution τ despite the largest σ (see. Table. 1) at least differentiates error of estimators. For small networks the mean value of the estimation error is less while using the estimators IDS or HU than LS. This is not observed in the case of large networks. For small networks and the gross error distributions f_1 or τ, the estimaton error of the robust methods is less dispersed around the mean then the error of LS. For distribution f_2 better results are obtained using HU than IDS. For large networks, there are no significant differences between the error of LS and the error of the robust methods.

8. Conclusions

- Small networks are strong – their median internal reliability indices are about 0.7. Large networks are weaker with median internal reliability indices about 0.4.
- For small networks the estimaton error of the robust methods is less than the error of LS, which confirms results in the literature. Moreover the Huber method has a smaller interquartile range than the iterative data snooping method.
- For large networks, there are no significant differences between the errors of LS, IDS and HU despite the fact that in considered error distributions outliers‘ probabilities are many times greater than for the normal distribution.
- Although the large networks are not strong, the average estimation error is comparable to the average error of the typical observations, so is acceptably small.

Acknowledgments
The author is greatly indebted to Piotr Pokarowski for his helpful suggestions which made this work possible.

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Appendix 1: Parameters of the test networks

siec_1, sigma0=0.08

	from	to	y	inverse weight
1	1	2	-485.27	2.0
2	1	7	1035.00	2.8
3	1	8	640.03	2.8
4	2	3	344.91	2.5
5	2	6	205.11	2.5
6	2	7	1519.91	4.0
7	2	8	1124.82	3.7
8	2	9	-629.92	2.5
9	3	4	565.07	1.8
10	3	5	919.79	3.7
11	3	6	-139.98	2.6
12	3	9	-975.12	2.8
13	3	10	1209.66	3.4
14	4	5	355.17	2.3
15	4	10	644.96	2.5
16	5	6	-1060.38	3.7
17	5	10	290.19	4.8
18	6	7	1315.31	4.4
19	7	8	-395.20	4.7
20	8	9	-1755.09	4.0
21	9	10	2185.07	3.7

siec_2, sigma0=0.08

	from	to	y	inverse weight
1	1	2	1.9	1
2	2	3	0.3	2
3	3	4	-0.7	1
4	4	5	-0.3	2
5	5	6	-2.1	2
6	6	7	-0.9	2
7	7	1	1.2	2
8	2	5	-0.1	4
9	5	7	-2.5	3
10	7	2	2.9	3
11	101	2	-1.0	2
12	1	102	3.8	5
13	102	103	-0.1	2
14	103	104	0.3	1
15	104	105	-0.3	1
16	105	102	0.1	4
17	102	101	-0.5	3
18	101	103	0.2	4
19	104	1	-4.4	4
20	7	105	4.8	3

Sw. Anna, sigma0=0.05

	from	to	y	inverse weight
1	104	102	0.50	2.0
2	102	103	-0.45	1.0
3	103	100	-0.06	3.0
4	100	3	0.00	3.0
5	3	2	0.11	2.0
6	2	106	0.07	1.0
7	2	1	0.05	1.0
8	1	6	-0.17	1.0
9	6	7	0.14	1.0
10	7	8	-0.10	1.0
11	8	9	0.08	1.0
12	9	10	0.14	1.0
13	10	113	-0.11	1.0
14	113	109	-0.07	1.0
15	109	108	0.07	1.0
16	108	107	-0.17	1.0
17	107	6	0.02	1.0
18	107	106	0.21	2.0
19	106	104	-0.17	3.0
20	10	11	-0.05	2.0
21	11	33	-0.05	1.0
22	10	37	0.02	1.5
23	37	36	-0.01	1.0
24	36	35	0.08	1.0
25	35	34	-0.03	1.0
26	34	33	-0.09	1.0
27	33	30	-0.95	2.0
28	30	52	0.52	3.0

Lazienki, sigma0=0.05

	from	to	y	inverse weight
1	14	100	0.42	2
2	100	101	-0.26	2
3	101	102	-0.24	1
4	102	103	-0.12	2
5	103	104	0.01	3
6	104	100	0.61	2
7	104	50	0.93	4
8	101	24	0.34	2
9	14	13	0.08	1
10	13	11	-0.06	1
11	11	12	0.05	1
12	12	25	-0.31	1
13	25	26	-1.29	1
14	26	27	0.87	1
15	27	28	0.47	1
16	28	29	-0.15	1
17	29	30	-0.01	1
18	30	31	-0.01	1
19	31	28	0.17	1
20	28	14	0.19	1
21	12	8	0.17	1
22	11	10	0.05	1
23	10	9	0.10	1
24	9	8	0.07	1
25	8	24	0.26	1
26	24	23	0.03	1
27	23	22	-0.20	1
28	22	21	0.87	2

29	52	51	0.01	1.0
30	51	31	0.27	2.0
31	31	32	0.03	1.0
32	32	35	0.24	2.0
33	1	4	-0.11	2.0
34	4	3	-0.05	1.0
35	4	15	1.49	3.0
36	15	16	-0.20	2.0
37	41	23	-0.96	1.0
38	16	17	-0.05	1.0
39	17	18	-0.67	1.0
40	16	22	-1.23	2.0
41	22	21	-0.22	1.0
42	22	221	-0.27	1.0
43	221	23	-0.09	1.0
44	23	24	-0.05	1.0
45	24	25	0.27	1.0
46	25	26	0.18	1.0
47	26	200	0.85	2.0
48	26	27	-0.14	1.0
49	27	28	0.24	1.0
50	28	29	-0.27	1.0
51	29	54	-0.82	1.0
52	54	53	0.69	1.0
53	53	30	-0.65	1.0
54	29	30	-0.78	1.0
55	22	29	-0.13	5.0
56	18	19	-0.07	1.0
57	19	20	-0.65	1.0
58	20	21	-0.01	2.0
59	22	15	1.43	4.0
60	43	42	-0.12	1.0
61	42	41	0.79	1.0
62	41	40	0.37	1.0
63	40	39	0.34	1.0
64	39	203	-0.99	1.0
65	203	3	-0.43	2.0
66	44	43	0.33	1.0
67	44	45	-0.23	1.0
68	45	46	0.28	2.0
69	43	56	-0.47	5.5
70	55	56	0.20	1.0
71	55	112	-0.26	5.0
72	111	110	0.20	1.0
73	110	46	0.13	4.0
74	111	112	-0.12	1.0
75	46	200	1.24	5.0
76	6	5	-0.17	2.0
77	5	12	0.52	3.0
78	12	13	-0.03	1.0
79	13	14	0.12	1.0
80	14	15	1.11	2.0

29	21	20	-0.37	1
30	20	19	0.14	1
31	19	18	-0.37	2
32	18	17	0.27	1
33	17	16	0.01	1
34	16	4	-0.45	1
35	8	71	-0.14	1
36	71	7	0.05	1
37	7	6	0.09	2
38	6	5	-0.07	1
39	5	4	0.26	1
40	4	3	-0.61	1
41	3	2	0.14	1
42	2	32	-2.13	1
43	2	34	-0.68	1
44	34	32	-1.45	2
45	2	1	0.15	2
46	1	15	-0.08	2
47	15	14	-0.03	1
48	14	54	0.45	1
49	54	53	-0.55	1
50	53	52	0.23	1
51	52	51	0.09	1
52	51	50	0.52	1
53	50	49	-0.30	1
54	49	48	-0.20	1
55	48	47	-0.26	1
56	47	46	0.11	1
57	46	45	0.18	1
58	45	54	0.18	1
59	45	44	-0.23	1
60	44	43	0.34	1
61	43	42	-0.77	1
62	42	41	-0.06	1
63	41	2	0.41	2
64	41	1	0.56	1
65	41	50	1.19	1