LEAST SQUARES COLLOCATION ALTERNATIVE TO HELMERT'S TRANSFORMATION WITH HAUSBRANDT'S POST – TRANSFORMATION CORRECTION

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Abstract

The paper presents a least squares collocation - based alternative to Helmert's transformation with Hausbrandt's post – transformation correction. The least squares collocation is used as an exact predictor i.e. it honors the data, thus the problem of zero residuals on transformation control points is overcome and zero residuals are assured by the method applied. Despite the fact that the procedure is presented for Helmert's transformation it may easily be copied to any other form of coordinate transformation. A numerical example is provided within the content of the paper.

Keywords: least squares collocation, Helmert's transformation, covariance function, post – transformation correction

1. Introduction

Coordinate transformation is a way of converting coordinates between two coordinate systems of which the first one is called a primary system and the second a secondary system. In practice, the basic process of coordinate transformation usually consists of two stages. In the first step a set of parameters of the transformation model (transformation function) is searched on the basis of homological points of the two systems. The solution to this problem is possible if the number of homological points is equal or greater than the number of transformation model parameters. The second step is performed for arbitrary points of the primary system. It relies on using the adopted transformation model to transform points from the primary system to the secondary one. The accuracy of the coordinate transformation is influenced by two factors. The first one is the type of the adopted transformation model (determined by the number of parameters). And the second one results from the limited accuracy of

the coordinates of homological points. The latter factor, in case of geodetic networks, is difficult to eliminate because it often results from measurements' uncertainty. Study on transformation model accuracy is possible only on homological points where the residuals are available (used as a measure of accuracy). The next stage in the process of coordinate transformation, called a post – transformation correction, may be the removal of residuals from control points. Computation of correction terms and their introduction to coordinates may increase the accuracy of the transformation process. The base for application of such post – transformation corrections are three assumptions:

- type of adopted transformation model is good enough and does not influence significantly the values of residuals
- coordinates of the secondary system are more accurate than those from the primary system
- residuals on a given control point represent also the uncertainty of coordinates within its neighborhood

The third from the above assumptions is the base for methods of computing post – transformation corrections as a function of the distance between a newly transformed point and control points – *Inverse Distance Method* (IDW). In this method the value of coordinate correction of an arbitrary point is inversely proportional to the distance d^k from control points, where exponent $k = 1..n$. The value of the exponent k is usually set empirically. In the well known method of Hausbrandt k is set to 2 (Hausbrandt, 1970).

   Although the procedure presented herein may be applied to any form of coordinate transformation (affine, projective, polynomial etc.) we limit ourselves only to the Helmert's transformation. The procedure may easily be copied for other coordinate transformations without any additional effort. Least squares collocation (here it is used in its simplest form without a functional analysis sound) may be viewed either as a filter (filters out the measurement error on observation points) or exact interpolator (ignores the fact whether a measurement error exists or not and honors the data by passing through observation points) depending on the model involved. The latter case seems to be a good choice when one wants to retain catalogue coordinates in the secondary coordinate system. This may be confronted with Hausbrandt's post – transformation correction which uses inverse distance interpolation method (IDW) to redistribute residuals from the transformation model onto newly transformed points. The Hausbrandt's method is based on a heuristic idea of removing residuals (obtained in the process of model adjustment) from catalogue coordinates of control points by making them all zeros. This as seems may be eliminated by a more objective numerical procedure with optimal properties – least squares collocation. By applying LSC as an exact interpolator one obtains the same values of coordinates in the secondary system as taken to create the transformation model and the residuals are redistributed in a signal-noise related part of the transformation model.

2. Helmert's transformation

Helmert's transformation also known as *four – parameter similarity transformation* or *2D linear conformal transformation* is probably the most used form of coordinate transformation or at least the most broadly described in the geodetic literature. In order to transfer points from one coordinate system to another it employs a scale

factor S, rotation angle α and translations T_X, T_Y. The basic equations (one for each coordinate) may be written as:

$$X = T_X + xS\cos\alpha + yS\sin\alpha \tag{1a}$$
$$Y = T_Y - xS\sin\alpha + yS\cos\alpha \tag{1b}$$

where:

X, Y – coordinates in the secondary system
x, y – coordinates in the primary system
T_X, T_Y – translations along X and Y axes, respectively
S – scale factor
α – rotation angle

Putting

$$a = S\cos\alpha \quad b = S\sin\alpha \tag{2}$$

(1a) and (1b) may be rewritten as:

$$X = T_X + ax + by \tag{3a}$$
$$Y = T_Y - bx + ay \tag{3b}$$

Adding error terms δ_X and δ_Y to (3a) and (3b); respectively (or assuming errors in both coordinate systems (δ_x and δ_y), (δ_X and δ_Y); respectively) and having more than the minimum number of observations (pseudoobservations) one obtains an adjustment problem to be solved with respect to the transformation parameters a, b, T_X, T_Y (Ghilani, 2010; Mikhail and Ackermann, 1976). After the adjustment process a scale factor and rotation angle may simply be restored by the formulas:

$$\alpha = a\tan\left(\frac{b}{a}\right) \tag{4}$$

$$S = \frac{a}{\cos\alpha} = \frac{b}{\sin\alpha} = \sqrt{a^2 + b^2} \tag{5}$$

3. Hausbrandt's post – transformation correction

In the process of adjustment of (3a, 3b), pseudo – observations (X and Y coordinates of control points in the secondary system) will be distorted by the values of residuals i.e. the catalogue (nominal) coordinates in the secondary system will be changed. In order to retain the nominal coordinates an artificial operation making residuals on control points zeros is undertaken and at the same time these residuals are redistributed on newly transformed points. This procedure was introduced by Hausbrandt, thus it carries his name – *Hausbrandt's post – transformation correction*. Redistributing procedure based on inverse distance interpolation due to Hausbrandt may be summarized as follows (Hausbrandt, 1970):

$$g_i = X_i^N - X_i^H \quad h_i = Y_i^N - Y_i^H \tag{6}$$

$$dX_P = \frac{\sum_i g_i w_i}{\sum_i w_i} \quad dY_P = \frac{\sum_i h_i w_i}{\sum_i w_i} \tag{7}$$

$$w_i = \frac{c}{\left(X_P - X_i\right)^2 + \left(Y_P - Y_i\right)^2} = \frac{c}{d_{iP}^2} \tag{8}$$

where:

X^N, Y^N – catalogue (nominal) coordinates in the secondary system
$X^{H'}$, Y^H – coordinates in the secondary system resulting from Helmerts's transformation
w – weights assigned to *g* and *h* based on inverse distance criterion
c – arbitrary constant

Distances d_{iP} are computed with the use of coordinates of points of the primary system. Corrections dX_P and dY_P are added to coordinates of newly transformed points according to (3a, 3b) with adjusted parameters *a*, *b*, T_X, T_Y.

4. Least squares collocation in brief

Least squares collocation (LSC) is an advanced technique of adjustment (adjustment, filtering and prediction) developed particularly for the research of the Earth's gravity field (*Physical Geodesy*) (Moritz, 1980, 1972; Krarup, 1969). As "advanced" often means "flexible" it quickly found its other applications. Mikhail reports its use in the field of photogrammetry (Mikhail and Ackermann, 1976; Kraus and Mikhail, 1972). Moritz (1972) gives a "user – friendly" review of applications to interpolation and transformation problems (he mentions also satellite observations and graduation errors of theodolite circles, and of course gravity measurements and combining heterogeneous measurements to describe the Earth's gravity field). Hardy (1977) shows (in fact the least squares prediction method which is a less general ancestor of LSC) its application to image processing. Thus, the scope of problems solved through the use of LSC is really broad. The basic equation of LSC (in a descriptive way) is expressed as:

$$\text{Observation = Trend + Disturbance = Trend + Signal + Noise} \tag{9}$$

In fact the most right hand side part of the above "equation", called *trend, signal and noise* model is the Henderson's *mixed effects model* (Henderson, 1950; Robinson, 1991). But here, it will be used in slightly modified form i.e. as an exact predictor thus the representation (middle part of the above) is closely related to that of Goldberger's BLUP or to the most often presented universal kriging variant (exact predictor) in geostatistics. In thus formulated problem we will not remove (filter out) a noise-related part of the model from observation points i.e. both signal and noise are combined in a disturbance term. Hence, the model (rewritten from (9) in a more formal way) is given as:

$$\mathbf{L} = \mathbf{Ax} + \mathbf{s} + \mathbf{n} = \mathbf{Ax} + \boldsymbol{\varepsilon} \tag{10}$$

where:

$\mathbf{L}$ – vector of observations
$\mathbf{A}$ – design matrix
$\mathbf{x}$ – vector of model parameters to be estimated
$\mathbf{s}$, $\mathbf{n}$, $\boldsymbol{\varepsilon}$ – vector of signal, noise, and disturbance; respectively

Derivation of model parameters is very elaborate thus only a summary will be presented here and the interested reader is referred to e.g. Moritz (1972), Goldberger (1962), Christensen (2011). The vector of estimates of **x** i.e. $\hat{\mathbf{x}}$ is given as:

$$\hat{\mathbf{x}} = \left(\mathbf{A}^T \mathbf{C}_{\varepsilon\varepsilon}^{-1} \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{C}_{\varepsilon\varepsilon}^{-1} \mathbf{L} \tag{11}$$

where:

$\mathbf{C}_{\varepsilon\varepsilon}$ – is the covariance matrix for the disturbance part of the assumed model (observables – observables, data – to – data)

One easily notices that (11) is the Aitken's estimator of generalized least squares method (BLUE – Best Linear Unbiased Estimator). The predicted disturbance term (BLUP – Best Linear Unbiased Prediction) is expressed as:

$$\hat{\boldsymbol{\varepsilon}}_0 = \mathbf{C}_{\varepsilon\varepsilon_0} \mathbf{C}_{\varepsilon\varepsilon}^{-1} \left(\mathbf{L} - \mathbf{A}\hat{\mathbf{x}}\right) \tag{12}$$

where:

$\mathbf{C}_{\varepsilon\varepsilon_0}$ – is the covariance matrix for the disturbance part of the assumed model (observables – nonobservables, data – to – target)
$\hat{\boldsymbol{\varepsilon}}_0$ – predicted disturbance

Total predicted value (estimated trend plus predicted disturbance term) may be written as:

$$\hat{\mathbf{L}}_0 = \mathbf{T}\hat{\mathbf{x}} + \hat{\boldsymbol{\varepsilon}}_0 = \mathbf{T}\hat{\mathbf{x}} + \mathbf{C}_{\varepsilon\varepsilon_0} \mathbf{C}_{\varepsilon\varepsilon}^{-1} \left(\mathbf{L} - \mathbf{A}\hat{\mathbf{x}}\right) \tag{13}$$

where:

T – is a design matrix for the newly predicted (here newly transformed) points according to (3a and 3b)

The error analysis may be summarized by three error matrices:

Error matrix for estimated parameters $\hat{\mathbf{x}}$:

$$\mathbf{E}_{\hat{\mathbf{x}}\hat{\mathbf{x}}} = \left(\mathbf{A}^T \mathbf{C}_{\varepsilon\varepsilon}^{-1} \mathbf{A}\right)^{-1} \tag{14}$$

Error matrix for predicted disturbance term:

$$\mathbf{E}_{\hat{\boldsymbol{\varepsilon}}_0 \hat{\boldsymbol{\varepsilon}}_0} = \mathbf{C}_{\varepsilon\varepsilon} - \mathbf{C}_{\varepsilon_0\varepsilon} \mathbf{C}_{\varepsilon\varepsilon}^{-1} \mathbf{C}_{\varepsilon\varepsilon_0} + \mathbf{H}\mathbf{A}\mathbf{E}_{\hat{\mathbf{x}}\hat{\mathbf{x}}} \mathbf{A}^T \mathbf{H}^T \tag{15}$$

where:

$$\mathbf{H} = \mathbf{C}_{\varepsilon_0\varepsilon} \mathbf{C}_{\varepsilon\varepsilon}^{-1}$$

Error matrix for the sum estimated trend + predicted disturbance (total prediction):

$$\mathbf{E}_{\hat{\mathbf{L}}_0 \hat{\mathbf{L}}_0} = \mathbf{C}_{\varepsilon\varepsilon} - \mathbf{C}_{\varepsilon_0\varepsilon} \mathbf{C}_{\varepsilon\varepsilon}^{-1} \mathbf{C}_{\varepsilon\varepsilon_0} + \left(\mathbf{H}\mathbf{A} - \mathbf{T}\right)\mathbf{E}_{\hat{\mathbf{x}}\hat{\mathbf{x}}} \left(\mathbf{A}^T \mathbf{H}^T - \mathbf{T}^T\right) \tag{16}$$

Issue of exactness
First, we shall sketch this graphically. In general case, when a new point to be predicted does not coincide with an observed one we have the structure stored in matrices $\mathbf{C}_{\varepsilon\varepsilon}$ and $\mathbf{C}_{\varepsilon_0\varepsilon}$ of the following form:
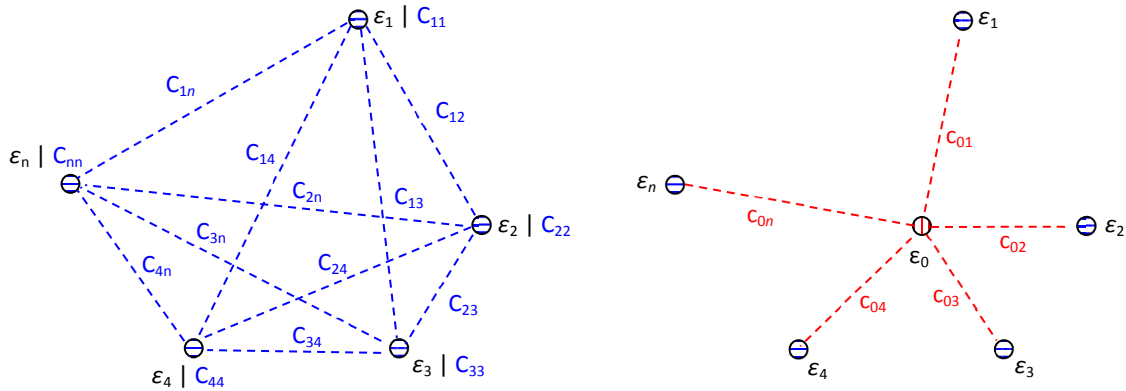
Fig. 1. Structure stored in the matrix $\mathbf{C_{\varepsilon\varepsilon}}$ (left drawing: covariance between each pair of observables) and the matrix $\mathbf{C_{\varepsilon_0\varepsilon}}$ (right drawing: covariance between each pair of observable – target, here it would be rather a vector than a matrix since only one target is to be predicted)

In the above case the structure stored in both covariance matrices is different, but this changes when a new point to be predicted coincide with an observed one. For the ease of presentation we will assume that this coincidence occurs for all observables i.e. all points to be predicted are located at observables. In this particular case both matrices $\mathbf{C_{\varepsilon\varepsilon}}$ and $\mathbf{C_{\varepsilon_0\varepsilon}}$ are the same.
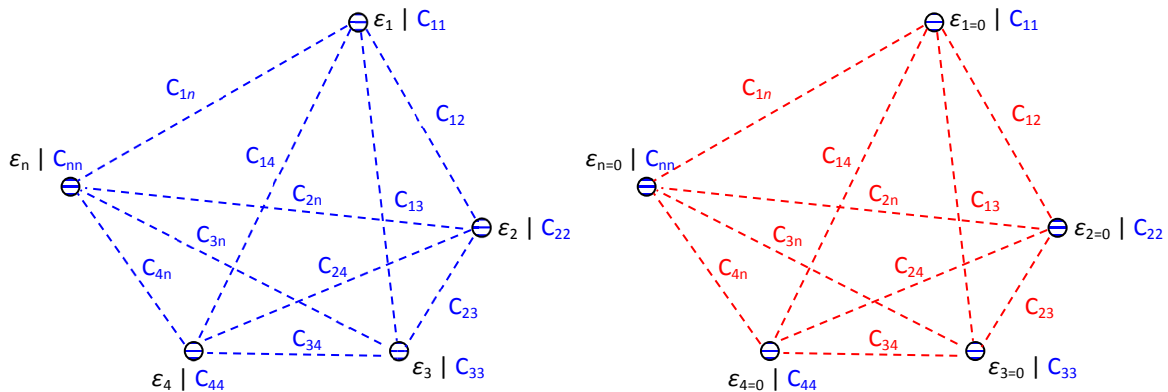

Fig. 2. Structure stored in the matrix $\mathbf{C_{\varepsilon\varepsilon}}$ (left drawing: covariance between each pair of observables) and the matrix $\mathbf{C_{\varepsilon_0\varepsilon}}$ (right drawing: covariance between each pair of observable – target, here target = observable)

Having in mind the above described (13) may be rewritten with ($\mathbf{T = A}$):

$$\hat{\mathbf{L}}_0 = \mathbf{A}\hat{\mathbf{x}} + \hat{\boldsymbol{\varepsilon}}_0 = \mathbf{A}\hat{\mathbf{x}} + \mathbf{C_{\varepsilon\varepsilon_0}}\mathbf{C_{\varepsilon\varepsilon}^{-1}}\left(\mathbf{L} - \mathbf{A}\hat{\mathbf{x}}\right) \qquad (17)$$

But as shown graphically both covariance matrices are equal in this case i.e. $\mathbf{C_{\varepsilon\varepsilon_0}} = \mathbf{C_{\varepsilon\varepsilon}}$, thus one finally obtains a simple relation:

$$\hat{\mathbf{L}}_0 = \mathbf{A}\hat{\mathbf{x}} + \mathbf{C_{\varepsilon\varepsilon}}\mathbf{C_{\varepsilon\varepsilon}^{-1}}\left(\mathbf{L} - \mathbf{A}\hat{\mathbf{x}}\right) = \mathbf{A}\hat{\mathbf{x}} + \mathbf{I}\left(\mathbf{L} - \mathbf{A}\hat{\mathbf{x}}\right) = \mathbf{L} \qquad (18)$$

This proves that this approach honors the data i.e. in case of (any) coordinate transformation catalogue coordinates of the secondary system will be retained.

5. LSC based Helmert's transformation procedure

In the LSC formalism (presented earlier), the input quantities for the transformation model to satisfy equations (11), (12) or (13) are as follows:

$$\mathbf{L} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_X \\ \mathbf{A}_Y \end{bmatrix} \quad \mathbf{T} = \begin{bmatrix} \mathbf{T}_X \\ \mathbf{T}_Y \end{bmatrix} \tag{19a, b, c}$$

$$\mathbf{C}_{\varepsilon\varepsilon} = \begin{bmatrix} \mathbf{C}_{XX} & \mathbf{C}_{XY} \\ \mathbf{C}_{YX} & \mathbf{C}_{YY} \end{bmatrix} \quad \mathbf{C}_{\varepsilon\varepsilon_0} = \begin{bmatrix} \mathbf{C}_{XX_0} & \mathbf{C}_{XY_0} \\ \mathbf{C}_{YX_0} & \mathbf{C}_{YY_0} \end{bmatrix} \tag{20a, b}$$

where:

> $\mathbf{L}$ – vector of observations (pseudoobservations, coordinates of control points in the secondary system, partitioned into blocks with respect of $\mathbf{X}$ and $\mathbf{Y}$ coordinates; respectively
>
> $\mathbf{A}$ – design matrix (pseudoobservations, function of coordinates of control points in the primary system, partitioned into blocks with respect of $\mathbf{x}$ and $\mathbf{y}$ coordinates; respectively)
>
> $\mathbf{T}$ – design matrix for newly transformed points (pseudoobservations, function of coordinates of control points in the primary system, partitioned into blocks with respect of $\mathbf{x}$ and $\mathbf{y}$ coordinates; respectively)
>
> $\mathbf{C}_{\varepsilon\varepsilon}$ – matrix of covariances and crosscovariances between residuals (on control points) based on a suitably selected covariance function, partitioned into blocks in relation to the partitioning of vector $\mathbf{L}$
>
> $\mathbf{C}_{\varepsilon\varepsilon_0}$ – matrix of covariances and crosscovariances between residuals (on control points and on points to be transformed) based on a suitably selected covariance function, partitioned into blocks in relation to the partitioning of vector $\mathbf{L}$

Subscripts XX, XY, YX, YY, XX_0, XY_0, YX_0, YY_0 denote the "direction" of covariance structure

The entire process may be summarized as:

1. Obtain a starting estimate of $\mathbf{x}$ by least squares, i.e. find $\hat{\mathbf{x}}$
2. Compute residuals from least squares adjustment, i.e. $\hat{\boldsymbol{\varepsilon}} = \mathbf{L} - \mathbf{A}\hat{\mathbf{x}}$
3. Produce an empirical covariance (cross – covariance) function of the residuals and fit a suitably selected theoretical model, on the basis of theoretical covariance (cross –covariance) function construct a corresponding covariance matrix, i.e. $\mathbf{C}_{\varepsilon\varepsilon}$
4. On the basis of the covariance matrix from step 3 obtain a new estimate of $\mathbf{x}$, i.e. $\hat{\mathbf{x}}$
5. Compute residuals from the generalized least squares adjustment, i.e. $\hat{\boldsymbol{\varepsilon}} = \mathbf{L} - \mathbf{A}\hat{\mathbf{x}}$
6. Repeat steps 3 – 5 until the relative or absolute change in estimates of $\hat{\mathbf{x}}$ is small
7. Construct the covariance matrix $\mathbf{C}_{\varepsilon\varepsilon_0}$
8. Make a final prediction according to (11), (12) and (13) and asses the accuracy of estimation and prediction process according to (14), (15), (16)

6. Numerical example

The following simple example is to show a step by step procedure how to solve the stated transformation problem with LSC (the authors are aware that every single surveyor is familiar with a part of computations presented below but they will be shown in a full extent to maintain the entire computation flowline). Along with the LSC solution, "classical" solution of Hausbrandt will also be computed and both sets of results confronted. The example is "tiny" on purpose, in order to be able to show all necessary intermediate steps of computations. The transformation problem consists of three control points (known coordinates in both primary (arbitrary) and secondary systems (arbitrary)) and three points to be transformed. All necessary data and a sketch of spatial distribution of points are presented below.
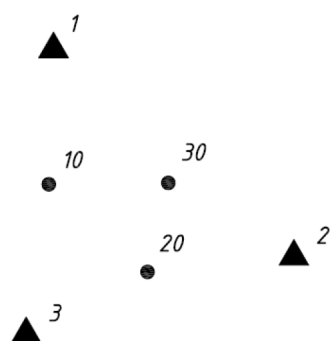
Fig. 3. Spatial distribution of points

Table 1. Control points and points to be transformed

Control points				
	Primary system		Secondary system	
No	X [m]	Y [m]	X [m]	Y [m]
1	14482.564	13288.071	5768950.542	6441593.071
2	8445.162	20281.612	5763055.723	6448708.668
3	6187.062	12491.598	5760639.634	6440965.177
Points to be transformed				
10	10550.348	13150.453		
20	8000.671	16023.344		
30	10591.893	16627.614		

Quantities common for both approaches (results presented without units since these are obvious.)

$$
L = \begin{bmatrix} 5768950.542 \\ 5763055.723 \\ 5760639.634 \\ 6441593.071 \\ 6448708.668 \\ 6440965.177 \end{bmatrix}
A = \begin{bmatrix} 14482.564 & 13288.071 & 1 & 0 \\ 8445.162 & 20281.612 & 1 & 0 \\ 6187.062 & 12491.598 & 1 & 0 \\ 13288.071 & -14482.564 & 0 & 1 \\ 20281.612 & -8445.162 & 0 & 1 \\ 12491.598 & -6187.062 & 0 & 1 \end{bmatrix}
T = \begin{bmatrix} 10550.348 & 13150.453 & 1 & 0 \\ 8000.671 & 16023.344 & 1 & 0 \\ 10591.893 & 16627.614 & 1 & 0 \\ 13150.453 & -10550.348 & 0 & 1 \\ 16023.344 & -8000.671 & 0 & 1 \\ 16627.614 & -10591.893 & 0 & 1 \end{bmatrix}
$$

L – vector of coordinates of the secondary system in the form of (18a)
A – design matrix of the transformation model of the form (18b)
T – design matrix for the newly transformed points of the form (18c)

Solution I: *Helmert's transformation with Hausbrandt's post – transformation correction*
The results will be presented in brief without error analysis.
Estimated transformation parameters (ordinary least squares) and vector of residuals:

$$\hat{\mathbf{x}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{L} = \begin{bmatrix} a \\ b \\ T_X \\ T_Y \end{bmatrix} = \begin{bmatrix} 0.999912 \\ 0.020311 \\ 5754199.364195 \\ 6428600.347024 \end{bmatrix} \qquad \hat{\boldsymbol{\varepsilon}} = \begin{bmatrix} \hat{\boldsymbol{\varepsilon}}_X \\ \hat{\boldsymbol{\varepsilon}}_Y \end{bmatrix} = \begin{bmatrix} -0.016 \\ -0.012 \\ 0.028 \\ -0.019 \\ 0.022 \\ -0.004 \end{bmatrix}$$

Coordinates of newly transformed (subscript *N*) points in the secondary system:

$$\mathbf{T}\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{X}_N \\ \mathbf{Y}_N \end{bmatrix} = \begin{bmatrix} 5765015.892 \\ 5762524.791 \\ 5765128.059 \\ 6441535.353 \\ 6444459.780 \\ 6445011.365 \end{bmatrix}$$

Hausbrandt's corrections according to (7) and final coordinates of the newly transformed points in the secondary system after Hausbrandt's post transformation correction (subscript *NH*) are given as:

$$\mathbf{D}_Y^X = \begin{bmatrix} \mathbf{dX} \\ \mathbf{dY} \end{bmatrix} = \begin{bmatrix} 0.001 \\ 0.006 \\ -0.004 \\ -0.007 \\ 0.004 \\ 0.004 \end{bmatrix} \qquad \mathbf{T}\hat{\mathbf{x}} + \mathbf{D}_Y^X = \begin{bmatrix} \mathbf{X}_{NH} \\ \mathbf{Y}_{NH} \end{bmatrix} = \begin{bmatrix} 5765015.893 \\ 5762524.797 \\ 5765128.055 \\ 6441535.346 \\ 6444459.784 \\ 6445011.369 \end{bmatrix}$$

Solution II: *Least squares collocation based Helmert's transformation*

We assume that a Gaussian covariance function model for the disturbances holds for both $\boldsymbol{\varepsilon}_X$ and $\boldsymbol{\varepsilon}_Y$, further we will only use *X* and *Y* subscripts.

$$C_{XX} = C_{YY} = C(c_0, c, a, d) = \begin{cases} c_0 + c & \text{if } d = 0 \\ ce^{-\left(\frac{d}{a}\right)^2} & \text{otherwise} \end{cases} \qquad \text{and} \qquad C_{XY} = C_{YX} = 0$$

where:
$c_0 = 0.00005$, $c = 0.0004$, $a = 6000$ m and d is a distance between points

We shall start from constructing distance matrix for the observables (control points) **D** and on its basis and the adopted covariance function a covariance matrix $\mathbf{C}_{XX} = \mathbf{C}_{YY}$ is obtained:

$$\mathbf{D} = \begin{bmatrix} 0.000 & 9240.163 & 8334.593 \\ 9240.163 & 0.000 & 8111.667 \\ 8334.593 & 8111.667 & 0.000 \end{bmatrix} \qquad \mathbf{C}_{XX} = \mathbf{C}_{YY} = \begin{bmatrix} 0.0004500 & 0.0000373 & 0.0000581 \\ 0.0000373 & 0.0004500 & 0.0000643 \\ 0.0000581 & 0.0000643 & 0.0004500 \end{bmatrix}$$

Cross – covariances are assumed to be zero in our simplified example, thus the total covariance matrix in division into blocks is of the following form:

$$\mathbf{C}_{\varepsilon\varepsilon} = \begin{bmatrix} \mathbf{C}_{XX} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{YY} \end{bmatrix}$$

Transformation parameters according to (11) are as follows:

$$\hat{\mathbf{x}} = \left(\mathbf{A}^T \mathbf{C}_{\varepsilon\varepsilon}^{-1} \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{C}_{\varepsilon\varepsilon}^{-1} \mathbf{L} = \begin{bmatrix} a \\ b \\ T_X \\ T_Y \end{bmatrix} = \begin{bmatrix} 0.999912 \\ 0.020311 \\ 5754199.367515 \\ 6428600.346875 \end{bmatrix}$$

Now, one may check numerically the exactness property of the presented method

$$\hat{\mathbf{L}}_0 = \mathbf{A}\hat{\mathbf{x}} + \mathbf{C}_{\varepsilon\varepsilon}\mathbf{C}_{\varepsilon\varepsilon}^{-1}\left(\mathbf{L} - \mathbf{A}\hat{\mathbf{x}}\right) = \begin{bmatrix} 5768950.542 \\ 5763055.723 \\ 5760639.634 \\ 6441593.071 \\ 6448708.668 \\ 6440965.177 \end{bmatrix} = \mathbf{L}$$

The above agrees with the input **L** vector.

In order to obtain the covariance matrix for observables – targets, we construct the distance matrix $\mathbf{D}_0$ first and entries of this matrix become the arguments of the covariance function what in the end results in a suitable covariance matrix. Both matrices are of the form:

$$\mathbf{D}_0 = \begin{bmatrix} 3935.074 & 7436.310 & 4413.245 \\ 7036.213 & 4281.932 & 3970.650 \\ 5127.974 & 4238.470 & 6042.979 \end{bmatrix} \quad \mathbf{C}_{XX_0} = \mathbf{C}_{YY_0} = \begin{bmatrix} 0.0002602 & 0.0000861 & 0.0002329 \\ 0.0001011 & 0.0002404 & 0.0002581 \\ 0.0001927 & 0.0002429 & 0.0001451 \end{bmatrix}$$

Since cross – covariances are assumed to be zero the total covariance matrix between observables and targets in division into blocks is of the following form:

$$\mathbf{C}_{\varepsilon\varepsilon_0} = \begin{bmatrix} \mathbf{C}_{XX_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{YY_0} \end{bmatrix}$$

Thus the final prediction gives the following results; the second term may be viewed as a correction term similar to that of Hausbrandt.

$$\hat{\mathbf{L}}_0 = \mathbf{T}\hat{\mathbf{x}} + \hat{\boldsymbol{\varepsilon}}_0 = \mathbf{T}\hat{\mathbf{x}} + \mathbf{C}_{\varepsilon\varepsilon_0}\mathbf{C}_{\varepsilon\varepsilon}^{-1}\left(\mathbf{L} - \mathbf{A}\hat{\mathbf{x}}\right) = \begin{bmatrix} 5765015.892 \\ 5762524.791 \\ 5765128.058 \\ 6441535.353 \\ 6444459.779 \\ 6445011.365 \end{bmatrix} + \begin{bmatrix} 0.003 \\ 0.007 \\ -0.005 \\ -0.010 \\ 0.006 \\ 0.003 \end{bmatrix} = \begin{bmatrix} 5765015.895 \\ 5762524.798 \\ 5765128.053 \\ 6441535.343 \\ 6444459.785 \\ 6445011.368 \end{bmatrix}$$

As a summary it is worth comparing two vectors of differences. The first one (on the left below) includes differences between predicted coordinates of newly transformed points by least squares collocation and coordinates of newly transformed points obtained by "classical" Helmert's transformation (based on least squares adjusted transformation model). The second vector (on the right below) includes differences between LSC newly transformed points and points transformed by means of Helmerts's transformation with Hausbrandt's post – transformation correction. All entries in both vectors are presented in millimeters.

$$\hat{\mathbf{L}}_0 - \mathbf{T}\hat{\mathbf{x}} = \begin{bmatrix} 3 \\ 7 \\ -6 \\ -10 \\ 5 \\ 3 \end{bmatrix} \qquad \hat{\mathbf{L}}_0 - \left(\mathbf{T}\hat{\mathbf{x}} + \mathbf{D}_Y^X\right) = \begin{bmatrix} 2 \\ 1 \\ -2 \\ -3 \\ 1 \\ -1 \end{bmatrix}$$

One may easily observe a decrease in differences what proves that LSC approach automatically "corrects" the transformation results. Now, one may argue which approach is better. Certainly, the Hausbrandt's approach or generally IDW based post – transformation corrections are easier both in theoretical and implementation terms. However, LSC approach has this advantage in the process of redistributing residuals that it includes the structure of residuals modeled by a covariance function not only the distance as it is in IDW approach.

7. Conclusions

In the paper a least squares collocation as an exact predictor has been applied to post – transformation correction as an alternative to the Hausbrandt's one. The LSC procedure is coherent in the sense that all the estimates (transformation parameters, predicted disturbances, error matrices) result from strict reasoning without a heuristic element. The application of LSC as an exact predictor automatically makes residuals on control points zeros since the adopted model honors the data. In the paper only a sketch of its application to the stated problem endowed with a simple numerical example is presented. It is definitely more laborious than Hausbrandt's approach as requires additional step involving covariance (also cross - covariance) structure modeling. It requires more data points in order to reasonably estimate an empirical covariance function. It provides a full and strict error analysis without any simplifications. In order to check its full practical potential it requires wider research conducted on field data and this is currently under the authors' scope.

References

Christensen R. (2011) Plane answers to complex questions: the theory of linear models, Springer

Ghilani C.D. (2010) Adjustment computations – spatial data analysis, 5th edition, Wiley, New Jersey

Goldberger A.S. (1962) Best Linear Unbiased Prediction in the Generalized Linear Regression Model, Journal of the American Statistical Association, 57(298), 369 – 375

Hardy R.L. (1977) Least squares prediction, Photogrammetric Engineering and Remote Sensing, 43(4), 475 – 492

Hausbrandt S. (1970) Rachunek wyrównawczy i obliczenia geodezyjne, PPWK Warszawa

Krarup T., (1969): *A contribution to the mathematical foundation of physical geodesy*, Geodaetisk Institut, Kobenhavn.

Kraus K., Mikhail E. M. (1972), Linear least – squares interpolation, Twelfth Congress of the International Society of Photogrammetry, Ottawa, Canada, July 23 – August 5, 1972, Presented paper, Commission III

Mikhail E.M. (with contributions by F. Ackermann) (1976), Observations and least squares, IEP – A Dun – Donnelley Publisher, New York

Moritz H. (1972) Advanced least – squares methods, Reports of the Department of Geodetic Science, Report No. 175, The Ohio State University

Moritz, H., (1980) Advanced Physical Geodesy, Herbert Wichmann Verlag, Karlsruhe,

Robinson G. K. (1991) That BLUP is a good thing: the estimation of random effects, Statistical Science, 6(1), 15 – 32

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