Reports on Geodesy and Geoinformatics, 2025, Vol. 120, pp. 86–95

DOI: 10.2478/rgg-2025-0019
Received: 13 August 2025 / Accepted: 3 December 2025
Published online: 15 December 2025

ORIGINAL ARTICLE

Application of the Simulated Annealing algorithm and robust weight functions for identification of constant reference points in 3D deformation analysis

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Abstract

Identification of the reference base in deformation analysis is a key element in obtaining the correct deformations of the monitored object. As it cannot be assumed that all the reference points are stable, a stability check must be conducted as the initial stage of the deformation analysis. Most known methods used for the identification of stable reference points cover leveling or horizontal networks. This study presents an approach using a coordinate transformation for the comparison of coordinates corresponding to two measurement epochs in 3D deformation analysis. The proposed algorithm is aimed at selecting such coordinate transformation parameters that correspond to the acceptable transformation residuals for the points of the reference base. Three robust adjustment weight functions (Kadaj, Huber, and Danish) were selected as a basis for formulating the objective function. The six 3D transformation parameters create a 6D search space. The optimization algorithm (simulated annealing) was applied for the search through the search space for desirable transformation parameters. The simulated example 3D two-epoch network was used for testing the performance of particular objective functions. The necessary parameters of the objective functions and optimization procedure were selected empirically. The test results confirmed the correctness of the adopted solution and the ability of all objective functions to detect the groups of fitting points. The objective function based on the Danish weight function appeared to be most promising due to the wide possibilities of shaping its features, which enable tailoring the objective function to the specific need.

Key words: stability analysis, 3D network, robust estimation, metaheuristic algorithm, simulated annealing

1 Introduction

Deformation analysis is an essential element in ensuring the safety of objects. It concerns engineering objects, residential environments, cultural heritage objects, and sometimes natural objects. There is a wide variety of technical means to get information about the object's movements. They can be divided into two groups. The first one can be described as geotechnical methods and includes a large set of devices measuring local changes in distances, tilts, tension, or water pressure (Chrzanowski, 1986; Moore et al., 2010; Maghsoudi and Kalantari, 2014). Geotechnical measuring devices are crucial for deformation monitoring systems and enable rapid detection of dangerous states of the object. However, due to the local character of the delivered information, the set of geotechnical sensors does not yield a general model of the changes in object geometry. Another group can be classified as geodetic methods. They deliver geometrical data concerning large distances and allow the connection of the object with the reference points that are not affected by the object deformations (Duffy et al., 2001; Taşçi, 2008; Henriques and Casaca, 2004; Karsznia et al., 2022).

Geodetic monitoring networks consist of measuring points that are connected by observations that are measured periodically. The single measurement of all necessary observations is called the measurement epoch. When the network consists only of points located on the object being monitored, it can deliver only information about

relative changes that can be used to determine the internal deformation of an object. Such information is often sufficient to assess the object's safety state.

When absolute deformations related to the constant reference system are required, the network must also contain constant reference points. The stability of the reference points must be maintained over the course of subsequent measuring epochs. Although the network designers make a big effort to design relevant locations and stabilization of the reference points, the actual stability of these points must be checked in each particular epoch.

Among the many methods used to identify stable reference points, two approaches can be distinguished. The first one considers identification as part of the process of determining displacements. The second approach lies in separation of identification as an independent process preceding the calculations of displacements (Prószyński and Kwaśniak, 2015).

In the first approach, the identification of an ultimate group of stable points usually runs iteratively and leads simultaneously to the identification of the reference system and the determination of ultimate displacement values of the object's points. Selected examples of this approach can be found in the literature (e.g., Amiri-Simkooei et al., 2017; Nowel and Kamiński, 2014; Baselga et al., 2015).

The second approach, stability analysis of reference points, is not connected with the determination of the object points' displacements. The stability analysis can use observations' values or point coordinates coming from subsequent epochs.

The main goal of the identification procedure is to select a group of reference points in which the internal geometry remains stable in the timespan between analysed epochs. This task can be challenging in practice, and the result can sometimes be equivocal. Except for point position uncertainties arising from inevitable measurement errors, the ground movements must be taken into consideration. It is more likely when the area the network covers is large and diverse in terms of geological structure (Prószyński, 2010; García-Asenjo et al., 2019). In such cases, it can be difficult to select a satisfactorily numerous group of reference points that show constancy of internal geometry features at the level corresponding with measurement precision (García-Asenjo et al., 2023). Another problematic situation can be the existence of more than one group of reference points that keep the mutual constancy of internal geometry. Most of the recently used algorithms lead to the selection of one dominating subgroup as a result. In most cases, it is satisfactory, but there are situations (e.g. geological faults) where the reference of deformations can be considered in many variants.

In this work, the author proposes a solution that is able to detect the subgroups of constant points among the potential reference points. The proposed algorithm is intended to detect multiple potential groups of mutually constant points when they exist. It uses the coordinates of reference points related to two epochs and a coordinate transformation as a way of matching two coordinate sets.

The main idea of the algorithm lies in searching through the transformation parameters to find such a transformation parameter set that corresponds to the acceptable fitting for the group of reference points. The special objective function (OF) measures the degree of fit between two sets of reference points. The set of transformation parameters constitutes the search space. Each point of the search space corresponds with the transformation and, in consequence, with the transformation residuals for the considered reference points. For further consideration, such a point in the search space will be called a P-set.

The level of complexity of such a formulated task depends mainly on the search space dimension, which arises from the type of transformation. Assuming the constant scale of the network, the search space dimension is 1 for the levelling network, 3 for the horizontal network, and 6 for the three-dimensional (3D) network.

Optimization algorithms are most appropriate for problems formulated in this way. There is a large family of optimization algorithms. Some of them, called metaheuristics, are inspired by natural processes or the behaviour of living organisms. They differ in computational efficiency, and the optimal choice depends on the number of sought parameters (search space dimension) and distribution of the OF values in the search space. Metaheuristic algorithms are commonly used in various areas of technology and science, including geodesy. 3D coordinate transformation (Odziemczyk, 2020), design of geodetic networks (Berne and Baselga, 2004; Odziemczyk, 2023; Mrówczyńska and Sztubecki, 2019; Doma and Sedeek, 2014; Yetkin, 2013), or adjustment of geodetic observations (Baselga, 2007; Mahboub et al., 2024) can be given as examples.

The idea of the algorithm proposed in this research has already been explored by the author in previous studies, as described in Odziemczyk (2021), and Odziemczyk (2025b), in the context of stability analysis of 2D networks. Increasing the search space dimension when applying the algorithm for 3D stability analysis results in an increase in the computation cost and the level of complexity of the OF distribution in the extended 6-dimensional search space.

For this reason, the simulated annealing (SA) optimization algorithm was applied. Its most important advantage is the simplicity and the lack of assumptions concerning the OF distribution.

Another important issue that must be addressed when applying optimization algorithms is the OF. The OF's arguments are the transformation parameters (P-set coordinates), and it returns a value that is the basis for the assessment of the current P-set. The minimum of the OF is usually associated with the sought solution. More details of the OF can be found in section 3.1.

The idea of this work was presented as a general outline in a conference presentation (Odziemczyk, 2025a). This paper presents it in more detail with an extended comparative analysis of OFs and stability assessment of the control points.

The article is organized in the following way. The background of the research, including a review of recently known deformation analysis methods and measurement techniques used for the acquisition of geometrical data, is presented in Section 2. Section 3 describes the idea of the method proposed in this work with its key elements: the OF and the SA algorithm. The numerical test of the proposed algorithm is described in Section 4. Section 5 contains the final conclusions, which are the results of the analyses conducted within the numerical test.

2 Related works

2.1 Deformation analysis methods

The problem of identifying stable reference points in deformation networks has a long history. The main factors affecting the development of identification methods are the development of measurement methods and the increase in computational efficiency.

The methods based on observation differences between two epochs have been commonly used for many years. The adjustment model uses point displacements as unknowns.

Such a model is simple, computationally efficient, and the differences of observations were free of some systematic measurement errors. This model could only be applied when the measurement repeated the same schema for all subsequent epochs, which was difficult (if possible) due to the changes appearing in the object area. Works of Nowel and Kamiński (2014) and Batilović et al. (2021) use a differential approach in deformation analysis.

The solution is the common adjustment of direct observations from two (or more) epochs (Brunner et al., 1981; Amiri-Simkooei et al., 2017). This approach must consider a larger number of observations and unknowns; it is more computationally costly, which used to make it less popular.

Both mentioned approaches use observations, and identification of stable reference points is usually conducted by iterative modification of the adjustment datum. The process finishes with the

selection of a datum composed of reference points for which their residual displacements can be considered insignificant. However, we cannot be sure that the identified subset of reference points is the only one for which insignificant residual displacements can be obtained.

An increase in computational power of available computers made it possible to apply methods that enable complex analysis of internal geometry of all reference points.

The combinatorial approach was proposed in Neitzel (2005) or Lehmann and Lösler (2017). Its idea lies in testing all possible combinations of reference points for their internal stability. The method is efficient in case stable points make a minority subgroup, even when the displacements of unstable points are relatively large. It must also be mentioned that the method can detect all potential subgroups of stable points. The high numerical cost of this method arises from a large number of combinations that have to be taken into consideration. This number grows exponentially with the network size and limits the method's applicability, even when using very fast modern computers.

The method proposed in this article uses the coordinates of reference points representing their positions in subsequent epochs. The coordinates are usually determined as a result of a single epoch preliminary adjustment, which also delivers the precision characteristics. The stability analysis has a form of searching for reference points adherence using coordinate transformation. Global Congruency Test (Niemeier, 1979; Denli and Deniz, 2003), Iterative Weighted Similarity Transformation (IWST) (Ambrožič et al., 2019; Chen, 1984), or the algorithm of numerical control of object adherence (Adamczewski, 1979) can be given as examples of this approach.

The essence of the above-mentioned methods is an estimation of the deformation model parameters using geodetic observations or coordinates. The fact that observations as well as coordinates are directly or indirectly affected by measurement errors causes the need for the application of an appropriate estimation method. Least Squares (LS) adjustment is the traditional algorithm applied for processing geodetic observations. However, in the newer works, the authors propose robust estimation methods as an alternative to LS. Their main advantage is the ability to determine the correct adjustment model parameters in case outliers exist in the observation set. Works of Nowel and Kamiński (2014); Duchnowski and Wiśniewski (2014); Batilović et al. (2021) are examples of this approach. In application to deformation monitoring, the robust estimation methods can be applied to establish a deformation model (Caspary and Borutta, 1987), including the identification of the deformation datum (Ambrožič et al., 2019).

Two optimization algorithms, SA and Hooke-Jeeves optimization, were used in the earlier works of the author (e.g., Odziemczyk, 2021, 2025b). 3D deformation analysis, which is the subject of this article, is a more challenging task because the sought solution is defined by six parameters. This is the reason for limiting the algorithm's choice to only one (SA), which is especially dedicated to complicated problems with a complex distribution of the OF. Three proposed variants of the OF are formulated using the weight functions (WF) of the known robust adjustment methods.

2.2 Measurement methods in 3D networks

Most deformation analyses in the area of geodetic measurements are performed in 2D (horizontal) or 1D (levelling) variants. It is mainly caused by the available measurement techniques. Linear, angular, or GNSS measurements can be used for horizontal displacements. Although GNSS basically delivers 3D coordinates (or coordinate differences) as a result, it does not translate into 3D deformation analysis. It mainly arises from the fact that the height coordinate precision is usually slightly worse than the horizontal one. Another factor that creates such a situation is the precise levelling technique, which is relatively simple to perform and, for not very large objects, delivers the precision unavailable for other methods.

There are, however, solutions where 3D analysis was performed. Denli and Deniz (2003) used GPS measurement results for the determination of 3D deformation of a geodetic network. A similar approach can be found in Fazilova and Sichugova (2021). Zhou et al. (2021) used GNSS and InSAR fusion for analysis of 3D deformations of the mining area. A very interesting example can be found in García-Asenjo et al. (2019); García-Asenjo et al. (2023), where precise linear measurements were used to establish a 3D geodetic deformation frame.

Finally, it is necessary to say that techniques like photogrammetry or terrestrial laser scanning are commonly applied to 3D deformation analysis, although they are mainly applied to smaller objects, applications like Wujanz et al. (2018), where the authors investigated ground movements.

3 Materials and methods

The stability analysis approach using the search space needs the formulation of the OF. The OF is defined for transformation residuals that correspond to the current transformation parameters (point in 6-dimensional search space). The selected optimization method can then be applied to search for the optimal solution. Both issues are discussed in the next two sections.

3.1 Objective function

The OF is expected to show the P-set that corresponds to such a transformation of a possibly large group of reference points that gives acceptably small transformation residuals for these points. In the case of constant points identification, the main features of the OF function are pointed out in Odziemczyk (2021). The properly selected OF should:

- return a significant signal value amplification for P-set corresponding to the group of reference points for which internal geometry is kept constant between the considered epochs,
- be insensitive to the points that moved between epochs (such points will obtain large transformation residuals),
- "promote" a larger size of the group of constant points.

The OF proposed in this research is based on the coordinate transformation, and it can be described by the following equation:

$$\mathbf{b}_i = \mathbf{R}\mathbf{a}_i + \mathbf{t} \tag{1}$$

where:

$\mathbf{a}_i$ – position vector for point i in epoch 1,
$\mathbf{b}_i$ – transformed position vector for point i,
$\mathbf{t}$ – translation vector,
$\mathbf{R}$ – rotation matrix.

In the case of 3D transformation, $\mathbf{R}$ is a 3x3 matrix and its elements are the functions of 3 independent rotation parameters. There are many definitions of rotation parameters used in various applications. For the needs of this study, the definition commonly used in photogrammetry (2) has been adopted:

$$\mathbf{R} = \begin{bmatrix} \cos\phi\cos\kappa & \cos\omega\sin\kappa + \sin\omega\sin\phi\cos\kappa & \sin\omega\sin\kappa - \cos\omega\sin\phi\cos\kappa \\ -\cos\phi\sin\kappa & \cos\omega\cos\kappa - \sin\omega\sin\phi\sin\kappa & \sin\omega\cos\kappa + \cos\omega\sin\phi\sin\kappa \\ \sin\phi & -\sin\omega\cos\phi & \cos\omega\cos\phi \end{bmatrix}. \tag{2}$$

Denoting $\mathbf{t} = \begin{bmatrix} \mathbf{t}_x & \mathbf{t}_y & \mathbf{t}_z \end{bmatrix}$ the transformation parameters vector (P-set) can be expressed as:

$$\mathbf{T} = \begin{bmatrix} \omega & \phi & \kappa & \mathbf{t}_x & \mathbf{t}_y & \mathbf{t}_z \end{bmatrix}. \tag{3}$$

The residual vector $\mathbf{r}_i$ for point i can be determined by equation (4):

$$\mathbf{r}_i = \mathbf{c}_i - \mathbf{b}_i \tag{4}$$

where $\mathbf{c}_i$ is the position vector of point i in epoch 2.

If there exists a group of points that keep their internal geometry in epochs 1 and 2, there exists the 3D transformation (P-set) resulting in acceptably small values of residual vectors $\mathbf{r}_i$ for these points.

Assuming both coordinate sets are unchanged during the identification process, the OF, which uses transformation residuals of the reference points, is a function of P-set coordinates.

Several attempts were made to define the most suitable OF. In Odziemczyk (2021) the sum of $|\mathbf{r}_i|$ was used for OF. To reduce the impact of the points with maximal $|\mathbf{r}_i|$ values, the sum was additionally reduced in case two or more points with insignificant residuals were found. Such an approach had to order all the considered points by $|\mathbf{r}_i|$ value, which increased the numerical cost of the operation. To simplify the way the OF is calculated, in Odziemczyk (2025b), the robust WFs were proposed. The same approach is continued in the current study.

Robust weight functions

Robust methods enable the estimation of parameters when outliers in observations can exist and were originally applied to the adjustment of geodetic networks. Many robust adjustment methods were developed. They are mostly based on iterative LS adjustment with the modified observation weights that are functions of adjustment residuals. The larger the adjustment residuum, the smaller the weight in the next iteration. The reweighing function is a specific feature of each adjustment method.

The general character of the robust weight functions that are sensitive to the small values and relatively insensitive to the larger ones coincides with the identification problem, in which we are looking for small transformation residuals for stable points, and we expect that large transformation residuals (unstable points) will not affect the solution. This concept was first used in Odziemczyk (2025b).

Three robust WF functions are applied in this study. The first one is the commonly known Huber function (Huber, 1964). It can be described by formula (5):

$$h(x) = \begin{cases} 1 & \text{if } \|\mathbf{x}\| < f \\ \frac{f}{\|\mathbf{x}\|} & \text{otherwise} \end{cases} \tag{5}$$

where $\|\mathbf{x}\|$ is the Euclidian norm of residual vector value. The f parameter stands for an observation's standard deviation. The Huber WF is quite simple, but it is composed of two elementary functions that require checking the argument value to choose the corresponding elementary function.

Another WF (6) was proposed by Kadaj (1978). k_x denotes here the standard deviation of x. In opposite to the Huber's function it is uniform in the whole range of argument values which simplifies the algorithmic side. It must be noted, however, that the function is more complex (it contains the power function):

$$k(x) = e^{-\frac{x^2}{2k_x^2}}. \tag{6}$$

The third WF comes from the Danish robust adjustment method (Krarup, 1980). It combines the features of both previously mentioned methods, but in contrast to them, it has as many as three control parameters:

$$d(x) = \begin{cases} 1 & \text{if } |x| < f \\ e^{-l|x-f|^{\lambda}} & \text{otherwise.} \end{cases} \tag{7}$$

Similarly to Huber's function, f is connected with the observation's standard deviation. Usually $\lambda = 2$ is taken for the network adjustment. l can vary in subsequent iterations and can be set to 0.01, 0.03, 0.05, 0.10.

The example distribution of the three selected WFs is shown in Figure 1.

The following parameters were assumed for the particular WFs:

$f = 7.0$ – for the Huber's WF,
$k_x = 16$ – for the Kadaj's WF,
$f = 7.0; \lambda = 0.75; l = 0.12$ – for the Danish WF.

The parameters for each WF are selected empirically. Their values are connected with network precision and the expected level of stability of the constant reference points. The above-listed parameter values were later used for the identification task described in Section 4.

It must also be noted that due to the three control parameters, the Danish WF is very flexible and can be shaped according to the particular need.

As can be seen in the Figure, all three robust WFs have a maximum of 1 for $x = 0$. For $|x| \gg 0$ the WF values decrease, tending asymptotically to 0. This feature can be applied to define the OF for the identification of stable reference points.

The three versions of the OF can be formulated using the three WFs defined with equations (5), (6), and (7):

$$F_H = \sum_{i=1}^{n} h\left(\|\mathbf{r}_i\|\right), \tag{8}$$

$$F_K = \sum_{i=1}^{n} \left[k\left(\|\mathbf{r}_i\|\right) - c\,\|\mathbf{r}_i\|^2\right], \tag{9}$$

$$F_D = \sum_{i=1}^{n} d\left(\|\mathbf{r}_i\|\right). \tag{10}$$

Each OF is calculated by summing a series of corresponding WFs for n reference points. The additional element in the Kadaj's OF arises from the fact that the $k(x)$ function (6) reaches its asymptotic values very fast (for $x/k_x = 9$, $k(x) = 0$ with numerical precision of 10^{-17}). When applied to constant points identification, it does not allow to distinguish between the P-sets that are far from the final solution. This fact can cause a problem for optimization algorithms that use the comparison of OF values to approach the solution. An element of the L2 norm causes that F_K gets different values even for large $\mathbf{r}$, for which $k\left(\|\mathbf{r}_i\|\right) = 0$. The value of the c parameter is selected empirically. It should be as small as possible to distinguish various solutions far from the optimum and not to disrupt the main OF distribution.

All three OFs return the maximal value for these P-sets that correspond to the congruent groups of reference points, for which $\|\mathbf{r}_i\|$ values are small. This means that P-sets with the maximal OF in the search space will be an object of interest for the search algorithm.

3.2 Stability assessment of the control points

Using the appropriately selected OF can lead to the detection of the places in the search space (P-sets) that correspond with the groups of congruent points. But except for the final P-set and corresponding value of the OF, the final result of the search must contain information on which points could be considered stable.

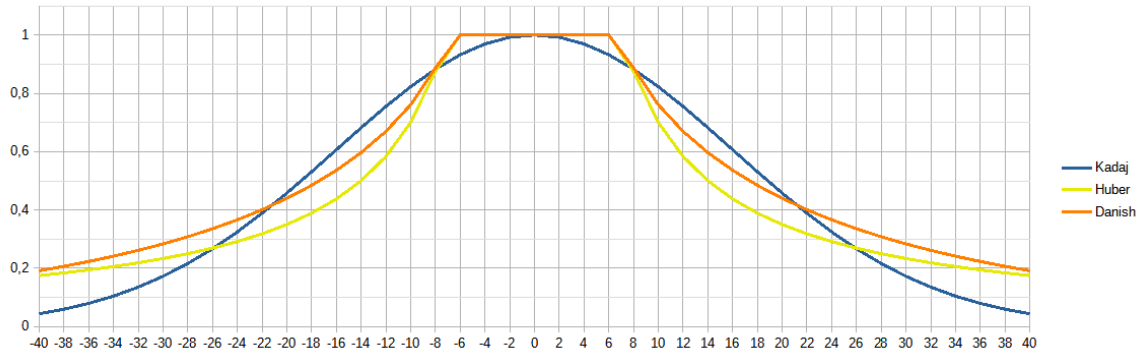

Figure 1. Distribution of the selected robust weight functions

For each P-set, the set of residual vectors $\mathbf{r}_i$ can be calculated. Each residual vector $\mathbf{r}_i$ is the indicator of the control points' stability. If the residual norm $\|\mathbf{r}_i\|$ will not exceed value resulting from measurement precision, the point can be assumed as stable:

$$\|\mathbf{r}_i\| < \varepsilon_i \tag{11}$$

As the point coordinates for both epochs come from preliminary adjustment, the positional precision can be used to formulate the threshold value:

$$\varepsilon_i = m\sqrt{\sigma^2_{Pi(1)} + \sigma^2_{Pi(2)}} \tag{12}$$

$$\sigma_{Pi(1)} = \sqrt{Q_{ix(1)} + Q_{iy(1)} + Q_{iz(1)}} \tag{13}$$

$$\sigma_{Pi(2)} = \sqrt{Q_{ix(2)} + Q_{iy(2)} + Q_{iz(2)}} \tag{14}$$

where: $Q_{ix(n)}$, $Q_{iy(n)}$, $Q_{iz(n)}$ — element of the covariance matrix for epoch n corresponding with x , y and z coordinates of point i.

Factor m denotes the relation between the precision characteristics resulting from the network measurement precision assessed by preliminary adjustments of both epochs and the final stability criterion arising from the expected level of points' stability.

It is initially set to 2, which corresponds to the situation when no detectable displacements of the reference points arose between the analysed epochs. In the case of a long period of time dividing the epochs, or when the terrain is not stable over a large area, we must be ready for the situation when even stable reference points will be affected by micro-displacements. The value of m has to be increased then, as described by García-Asenjo et al. (2023).

The final task of the reference system identification is to find such transformation parameters represented by **R** and **t** that the residual vectors for the most numerous group of potentially stable reference points will not exceed the assumed threshold.

3.3 Simulated annealing algorithm

As it was mentioned in the introduction, there are many optimization algorithms that could be taken into consideration when searching through the solution space. In the works of Odziemczyk (2021, 2025b), SA, Hooke-Jeeves, and hybrid algorithms were used. Both mentioned publications deal, however, with the 2D network where the search space has three dimensions (two components of translation and one rotation angle). The current research concerns 3D space, which results in a 6D search space in which the OF distribution is much more complex. That is the reason why the Hooke-Jeeves algorithm, which appeared to be more or less effective in the case of a 2D network, was not taken into account in recent research. Considering the multidimensional search space with a difficult-to-predict OF distribution, the SA algorithm was chosen. It was

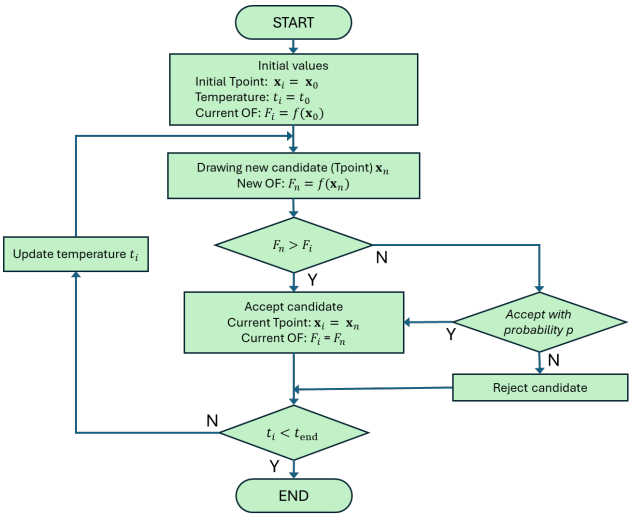

Figure 2. Flowchart of the SA algorithm

proposed in 1953 by Metropolis et al. (1953) and developed by Kirkpatrick et al. (1983). It is a convenient choice due to its ability to skip local minima if they exist.

The flowchart of the algorithm can be seen in Figure 2. It must be noted that this version of the algorithm is aimed at the search for the maximum of the OF.

The basic idea of the SA algorithm is iterative searching through the search space, looking for the best solution (P-set) reflected by the optimal (in this study, maximal) value of the OF. The new P-set is generated randomly using a normal distribution centred at the current P-set. The standard deviation of the applied normal distribution is connected with the temperature, and it decreases with subsequent iterations. Assuming $\mathbf{x}_i$ is a current P-set, $\Delta\mathbf{x}_i$ is a randomly generated incremental vector to the particular components of the new, potentially better, P-set, the relation (15) describes the way of choosing a new P-set:

$$\mathbf{x}_{i+1} = \begin{cases} \mathbf{x}_i + \Delta\mathbf{x}_i & \text{if } f(\mathbf{x}_i + \Delta\mathbf{x}_i) > f(\mathbf{x}_i), \\ \mathbf{x}_i + \Delta\mathbf{x}_i & \text{if } f(\mathbf{x}_i + \Delta\mathbf{x}_i) \le f(\mathbf{x}_i) \text{ with probability } p, \\ \mathbf{x}_i & \text{otherwise.} \end{cases} \tag{15}$$

The new solution is generally the one corresponding to the higher value of the OF (case 1).

The SA algorithm takes into account the possibility of accepting the new solution, which is worse than the current one (case 2). It can be useful in cases of an extremely complex distribution of the OF in the search space when there are many local minima, and the

global minimum is not clearly distinguishable. In the current study, this option is not considered, so the probability p is set to 0.

Several key issues must be specified for the application of the SA method to the particular task:

Definition of the objective function

The OF defines the search task and evaluates the current P-set, allowing the approach to the solution. In this work, the OF is described in Section 3.1.

Cooling scheme

The SA algorithm is inspired by the natural process of liquid solidification, which leads to a crystalline network of the lowest possible energy. The temperature reflects the average energy of the particle, and it decreases with time. The applicational temperature parameter decides on the scope (standard deviation) of the search for a new P-set ($\Delta\mathbf{x}_i$). The cooling scheme includes the initial temperature and the formula for the decrease in temperature with subsequent iterations. The separate factor t is assumed to be a temperature in this application. $t_0 = 1$ was taken as the initial value of t. The change in t values in subsequent iterations is described with the formula (16:

$$t_i = t_{i-1}\beta - t_0\beta \tag{16}$$

where i denotes the iteration number, and β is the cooling factor. Its value is selected depending on the nature of the optimization problem.

Molecule movement model

The molecule movement model consists of two elements. The first one is the way of generating a new solution. In most applications, a new solution is generated as it is described by the formula (17):

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \Delta\mathbf{x}_i. \tag{17}$$

The P-set components constitute vector $\mathbf{x}$. Incremental vector $\Delta\mathbf{x}_i$ is generated randomly, using normal distribution $N(0, \sigma_i)$, where σ_i is the vector of standard deviations of particular transformation parameters which are components of $\mathbf{x}$ in i-th iteration. The value of the standard deviation σ_i in iteration i is defined by formula (18:

$$\sigma_i = \sigma_0 t_i = \sigma_0 t_0 \beta^i. \tag{18}$$

The initial values of the standard deviation σ_0 are selected depending on the nature of the problem, and they reflect the assumed range of the corresponding parameter search scope. As the $\mathbf{x}$ components are differentiated in terms of character and units, it is important that the corresponding components of σ_0 were properly harmonized.

The acceptance criterion for a new solution is the second element of the molecule movement model. It is defined by the formula (15) with $p = 0$, which means there is no possibility of accepting a worse solution than the current one.

Criterion for finishing the iterative process

There are several possibilities for finishing the iterative process. The relevant criterion can be formulated based on:

- the final temperature,
- the number of iterations,
- the acceptable value of the OF.

In this study, temperature threshold t_{end} is applied (19):

$$t_i < t_{end} \tag{19}$$

In contrast to the parameters like σ_0 or β that are connected with the assumed points' stability, t_{end} determines the expected precision of the final solution. The value $\sigma_0 t_{end}$ (Eq. 18) defines the minimal search extent for the SA algorithm.

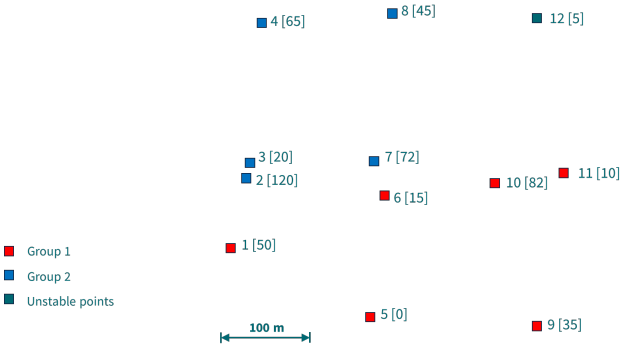

Figure 3. The test network

As mentioned in the introduction, the main task of the proposed algorithm is to find all potential subgroups of the reference points. Each subgroup is reflected by the corresponding minimum of the OF in the search space. Although the SA algorithm is basically dedicated to finding the global minimum in cases where local minima exist, the local minima are also of interest. By controlling the cooling factor, we can control the SA algorithm's performance. When the cooling factor β is closer to 1, the SA algorithm will be more likely to find the global minimum. Decreasing the cooling factor makes the algorithm faster and more likely to finish the search process in the local minimum corresponding to the smaller groups of stable reference points.

The application of the SA algorithm to the stability analysis of the reference points needs multiple runs of the search procedure. The initial P-set is each time selected randomly, which increases the probability of finding another independent solution if it exists.

The selection of the optimal value of the cooling factor is crucial for this approach. Too fast cooling leads to detecting the local minima, which are junk solutions that do not correspond with the acceptable subgroups, the real subgroups of the stable control points. When the cooling is too slow, only the global minimum can be found, which is not always desirable. At the current state of the research, the empirical approach is the most advisable. The algorithm was implemented as Python code using the Spyder programming environment.

4 Numerical test

The performance of the proposed method was tested on the simulated example network. The network consists of 12 points with 3D coordinates representing two epochs. Figure 3 shows the points' placement. The values in square brackets are Z coordinates. All the points' coordinates are shown in Table 1. The coordinates of epoch 2 were prepared in such a way that network points make two internally stable groups. One point does not belong to any group. For each point and each epoch, the raw coordinates were noised by random errors with the normal distribution and standard deviation $\sigma_X = \sigma_Y = \sigma_Z = 2$ mm. The random errors were generated using the Box–Muller transform.

The positional standard deviation for each point and each epoch is:

$$\sigma_{Pi(1)} = \sigma_{Pi(2)} = \sqrt{\sigma_X^2 + \sigma_Y^2 + \sigma_Z^2} = 3.5 \text{ mm}$$

Using equation (12), for $m = 2$, the stability criterion is $\varepsilon_i = 10$ mm.

Table 1. Coordinates of points of the test network for both epochs

No.	Epoch 1			Epoch 2			Differences		
	X [m]	Y [m]	Z [m]	X [m]	Y [m]	Z [m]	ΔX [mm]	ΔY [mm]	ΔZ [mm]
1	150.084	14.878	50.294	150.061	14.881	50.317	-23	3	23
2	230.056	29.801	120.343	230.074	29.764	120.378	18	-37	35
3	245.105	34.866	20.353	245.127	34.828	20.387	22	-38	34
4	410.089	49.777	65.442	410.112	49.736	65.486	23	-41	44
5	70.162	174.941	0.370	70.153	174.953	0.388	-9	12	18
6	210.160	189.882	15.446	210.146	189.885	15.476	-14	3	30
7	250.130	179.828	72.458	250.148	179.787	72.501	18	-41	43
8	420.150	199.788	45.552	420.170	199.748	45.602	20	-40	50
9	60.213	369.919	35.503	60.204	369.929	35.525	-9	10	22
10	225.174	319.829	82.545	225.166	319.827	82.578	-8	-2	33
11	235.235	399.877	10.606	235.229	399.880	10.635	-6	3	29
12	415.226	364.818	5.667	415.256	364.794	5.672	30	-24	5

Four variants of the OF were used in the numerical test:

i. Kadaj with k = 16 mm, and c = 0.1,
ii. Huber with f = 7 mm,
iii. Danish 1 with f = 7 mm, l = 0.12, and λ = 0.75,
iv. Danish 2 with f = 7 mm, l = 0.15, and λ = 0.85.

The above-listed parameters for each OF were selected experimentally. The initial values of k (for Kadaj OF) and f (for Huber and Danish OFs) were formulated as follows:

$$f = \sqrt{\sigma^2_{Pi(1)} + \sigma^2_{Pi(2)}} \tag{20}$$

$$k = 2\sqrt{\sigma^2_{Pi(1)} + \sigma^2_{Pi(2)}} \tag{21}$$

The above values were later modified in such a way that the algorithm delivers the largest possible group of stable points. We must remember that the final assessment of the stability that uses equation (11) is independent of the search procedure.

The search range for SA algorithm (σ_0) was selected as follows:

$$\sigma_0 = \begin{bmatrix} 0.05 & 0.05 & 0.05 & 0.005 & 0.005 & 0.005 \end{bmatrix}$$

The first three components are in [m] and they correspond to translation ($\mathbf{t}_x$ $\mathbf{t}_y$ $\mathbf{t}_z$). The components 4-6 correspond with rotation (ω ϕ κ) and are in [gon].

The value of the translation components is conditioned by the estimated limit of the points' displacements, and the relation between translation and rotation parameters depends on the network extent. In this example, both values were assumed by the creation of the numerical data.

β = 0.9995 was taken as the SA cooling factor for all variants of the OF. A 500-times run for each objective function was applied to detect the potential subgroups of the reference points. As it was mentioned, a random starting point was selected for each run.

To show the convergence character of the optimization process, the example of a single run was presented in Table 2. The particular rows correspond to the iterations when the improvement in OF occurred. The individual columns show: iteration number, the new transformation parameters (P-set coordinates), and the new value of the OF. Danish 2 OF was used in the example.

The last stage of each run was selection of the points that will be considered stable. The example presented in Table 2 resulted in the selection of group 1. Table 3 shows the complete test results.

As one can see, all three OFs appeared to be able to correctly identify both groups of the constant points. As could be expected, the first, larger group, was more likely to be detected by any of the OF. Except for Danish 1, in all cases, a certain number of procedure runs delivered results that are not identical with either of the assumed groups. In most cases, they were subsets of the assumed groups, pairs of points not belonging to the same group, or single points. The number of those "junk" solutions did not exceed 5% of runs.

It can also be observed that the probability of detecting particular groups varies depending on the applied OF. The Kadaj and Danish 1 OFs strongly preferred the larger subgroup. The Danish 1 is also the only OF that was able to avoid "junk" solutions.

As the main objective of the analysis is the detection of all existing subgroups of the reference points, we will consider more desirable those OFs that allow us to identify the minor subgroups with reasonable probability. In the case of the above example, the results of Huber and Danish 2 OFs are most advantageous. When using OFs preferring the dominating minimum in the search space, and in some less clear cases, the minor subgroups can be identified with even lower probability, and they can be easily missed.

5 Conclusions

The proposed algorithm enables the identification of the constant reference points in 3D networks. The increase in the network dimension by 1 leads to an increase in the search space by 3. In consequence, the search task is no longer as clear and intuitive as it was in the case of a 2D network. That is the reason why the optimization methods that were applicable for 2D appeared to be useless for the 3D task.

The simulated annealing (SA) optimization algorithm applied in the research is simple and can work with no restrictions concerning the OF distribution. Its only disadvantage was high requirements for computational power, which in this research was escalated by increased search space dimension and a slow programming environment (Python and Spyder). For professional applications, it is advisable to use a compiled version of the software (exe file).

The conducted tests confirmed that all three OFs, with the application of the SA algorithm, are able to identify the congruent subgroups that can create a reference base for the 3D displacement monitoring networks.

The proposed objective functions properly distinguish groups of stable points, although the probability of detecting particular groups varies depending on the applied OF. It must be noted that the Danish WF-based OF, due to three parameters, is the most flexible and can be easily adapted to specific cases.

Although the SA algorithm proved to be sufficiently effective and was able to reach the optimal P-set in the search space corresponding to the acceptable group of stable reference points, in some cases, it may deliver incorrect solutions. In such cases, a filtering procedure is necessary. Such a procedure should unify the solutions when one is a subset of another. The solutions containing fewer than three constant points or solutions with OF value significantly smaller than the OF value of the dominating solutions can be eliminated.

Table 2. Example of single optimization process convergence

It. No	t_x	t_y	t_z	ω	φ	κ	OF
0	-0.015	0.018	0.005	0.0026	-0.0089	-0.0093	0.605
23	-0.008	-0.048	0.066	-0.0056	-0.0051	-0.0074	0.930
41	0.013	-0.037	0.040	-0.0044	-0.0030	-0.0094	1.851
82	-0.019	-0.009	0.035	-0.0039	-0.0040	0.0025	2.051
179	-0.022	-0.007	0.006	-0.0060	0.0011	0.0076	2.129
255	-0.005	-0.003	0.039	-0.0043	0.0011	0.0038	2.571
318	0.017	-0.038	0.046	-0.0060	0.0049	0.0005	4.064
1266	-0.009	0.002	0.032	-0.0024	0.0075	0.0025	4.418
1268	-0.013	0.003	0.037	0.0012	0.0064	0.0047	4.749
1710	-0.014	-0.004	0.032	-0.0004	0.0080	0.0035	4.963
1856	-0.017	0.000	0.034	-0.0032	0.0072	0.0029	5.060
2597	-0.011	-0.001	0.038	-0.0019	0.0074	0.0035	5.243
2629	-0.013	0.003	0.036	-0.0001	0.0043	0.0036	5.481
2730	-0.011	0.001	0.031	-0.0005	0.0037	0.0038	5.544
2846	-0.017	0.000	0.032	0.0010	0.0040	0.0030	5.779
3055	-0.015	-0.001	0.031	0.0002	0.0033	0.0023	5.867
3164	-0.014	0.000	0.028	-0.0007	0.0034	0.0023	6.115
3502	-0.011	0.002	0.030	-0.0002	0.0040	0.0022	6.122
4363	-0.010	0.001	0.029	-0.0002	0.0033	0.0024	6.136
5272	-0.011	-0.001	0.029	-0.0003	0.0029	0.0024	6.140
5674	-0.010	0.000	0.029	-0.0001	0.0028	0.0026	6.142
5900	-0.011	-0.001	0.029	-0.0001	0.0029	0.0029	6.143
6530	-0.011	-0.001	0.029	-0.0001	0.0027	0.0031	6.143
7308	-0.011	-0.001	0.029	0.0000	0.0029	0.0030	6.144
7805	-0.011	-0.001	0.029	0.0001	0.0029	0.0029	6.144

Table 3. Test results

reference base	Kadaj		Huber		Danish 1		Danish 2	
	OF	Hits	OF	Hits	OF	Hits	OF	Hits
1 5 6 9 10 11	5.86	78.0%	6.83	68.8%	6.78	82.4%	6.14	60.0%
2 3 4 7 8	4.90	18.6%	5.97	30.4%	5.91	17.6%	5.18	35.4%
other	<4.49	3.4%	<5.20	0.8%	-	-	<3.89	4.6%

The optimal OF parameters must be selected empirically in the current study. Further research can be aimed at increasing the level of automation of the proposed algorithm. Formulating the rules for selecting the OF parameters can use preliminary adjustment, accuracy parameters, or historical data of points' movements. Although the SA algorithm appeared to be efficient enough, there is another possible direction for future studies, i.e., testing of other optimization methods and selecting their parameters.

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